

# A Luttinger's theorem revisited

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For uniform systems of spin-less fermions in  $d$  spatial dimensions with  $d > 1$ , interacting through the isotropic two-body potential  $v(\mathbf{r} - \mathbf{r}')$ , a celebrated theorem due to Luttinger (1961) states that under the *assumption* of validity of the many-body perturbation theory the self-energy  $\Sigma(k; \varepsilon)$ , with  $0 \leq k \lesssim 3k_F$  (where  $k_F$  stands for the Fermi wavenumber), satisfies the following universal asymptotic relation as  $\varepsilon$  approaches the Fermi energy  $\varepsilon_F$ :  $\text{Im}\Sigma(k; \varepsilon) \sim \mp \alpha_k (\varepsilon - \varepsilon_F)^2$ ,  $\varepsilon \gtrless \varepsilon_F$ , with  $\alpha_k \geq 0$ . As this is, by definition, specific to self-energies of Landau Fermi-liquid systems, treatment of non-Fermi-liquid systems are therefore thought to lie outside the domain of applicability of the many-body perturbation theory; that, for these systems, the many-body perturbation theory should *necessarily* break down. We demonstrate that  $\text{Im}\Sigma(k; \varepsilon) \sim \mp \alpha_k (\varepsilon - \varepsilon_F)^2$ ,  $\varepsilon \gtrless \varepsilon_F$ , is *implicit* in Luttinger's proof and that, for  $d > 1$ , in principle nothing prohibits a non-Fermi-liquid-type (and, in particular Luttinger-liquid-type)  $\Sigma(k; \varepsilon)$  from being obtained within the framework of the many-body perturbation theory. We in addition indicate how seemingly innocuous Taylor expansions of the self-energy with respect to  $k$ ,  $\varepsilon$  or both amount to tacitly assuming that the metallic system under consideration is a Fermi liquid, whether the self-energy is calculated perturbatively or otherwise. Proofs that a certain metallic system is in a Fermi-liquid state, based on such expansions, are therefore tautologies.

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## I. INTRODUCTION

Landau's (1957) phenomenological theory for the low-lying excited states of many-particle systems is based on the *assumption* that these states are characterised by a distribution of a dilute gas of quasi-particles (QPs), implying that interaction amongst these QPs may, to a good approximation, be neglected (Nozières 1964, Pines and Nozières 1966, Abrikosov, Gorkov and Dzyaloshinski 1975) (for a comprehensive review see Platzman and Wolff (1973)). <sup>1</sup> This ideal situation corresponds to the case in which QPs are the *exact* one-particle eigenstates of the many-body Hamiltonian, or, equivalently, the energies of the QPs are infinitely sharply defined. The success of Sommerfeld's independent-particle model (for example Ashcroft and Mermin (1981, Ch. 2)) for conduction electrons in simple metals, despite the apparent non-negligible strength of the electron-electron Coulomb interaction, can be viewed as a most strong pillar of Landau's phenomenological theory. Incorporation of the residual two-body interaction among the ideal QPs (at least beyond the Hartree-Fock or the *static* exchange approximation), which evidently should be weaker than the bare Coulomb interaction (insofar as low-energy scattering processes are concerned), <sup>2</sup> renders the QP states non-stationary. In other words, the actual QPs, if such entities can at all be meaningfully defined (§ III), are merely approximate one-particle eigenstates of the many-body Hamiltonian; the non-stationary nature of a QP 'eigenstate' signifies that its energy, in contrast with that of an exact eigenstate, is un-sharp, or the spectral function

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<sup>1</sup> To be precise, Landau's Fermi-liquid theory is applicable for temperatures  $T$  in the range  $(T_c, T^*)$ , where  $T^*$  and  $T_c$  stand, respectively, for the *coherence* and the *transition* temperature; for  $T$  below  $T_c$  the system is in a superconducting state (Kohn and Luttinger 1965). For details as well as a comprehensive discussion of instabilities of Fermi liquids see (Metzner, Castellani and Di Castro 1998).

<sup>2</sup> Otherwise the independent-particle model would not be an appropriate zeroth-order approximation.

of QPs consists of peaks with finite widths. Thus the true, that is interacting, QPs correspond to superpositions of nearly degenerate (as well as degenerate) many-body eigenstates, so that an initially well-defined QP ceases to be particle-like (in the sense of possessing a reasonably well-defined energy) with the passage of time.

The information with regard to QPs is contained in the single-particle Green function  $G$ . The formal Lehmann (1954) spectral representation of this function in the energy domain reveals that energies of the single-particle excitations, the QPs, can be identified with the singular points (often inappropriately designated as ‘poles’ — § V) along the real energy axis ( $\varepsilon$ -axis) (Fetter and Walecka 1971). Such observation is of little practical relevance when systems in the thermodynamic limit are concerned. For *macroscopic* systems, the continuum of the energy levels<sup>3</sup> gives rise to branch points (which are not *isolated* singularities) and branch cuts along the energy axis. This is naturally consistent with the above-indicated observation with regard to the broad rather than delta-function-like spectral function of QPs. Such a dramatic change in the analytic structure of  $G(\varepsilon)$  gives rise to a number of effects the subtleties of which are often overlooked in the literature. In this work we discuss in some detail a number of these, insofar as they are relevant to the main objective of our work. One of these effects is associated with the fact that whereas a reasonably sharp peak in the spectral function may be described by a single pole with some small imaginary part, such description is in violation of the fact that  $G(\varepsilon)$  is a perfectly regular function on the entire complex energy plane, excluding the real  $\varepsilon$ -axis. In § III we show that such complex poles do *not* simply correspond to the analytic continuation of  $G(\varepsilon)$  into the complex energy plane (i.e. the physical Riemann sheet (RS)), rather to the analytic continuation of the latter to a RS; we refer to this as a non-physical RS (see Appendix A).

The main complication, from the restricted viewpoint of the present paper, arises from the branch points of the Green function  $G(k; \varepsilon)$ , in particular that at  $\varepsilon = \varepsilon_F$  which can be shown to be also a branch point of the self-energy,  $\Sigma(k; \varepsilon)$ . From this it follows that, for instance, such expression as (Hugenholtz 1957, DuBois 1959b, Luttinger 1961)  $\text{Im}\Sigma(k; \varepsilon) \sim \mp \alpha_k (\varepsilon - \varepsilon_F)^2$ ,  $\varepsilon \gtrless \varepsilon_F$ , *cannot* be considered as the leading term in the Taylor series expansion of the self-energy operator at  $\varepsilon = \varepsilon_F$ . Thus there exists no *a priori* reason for the dispersion of the QP energies<sup>4</sup> to be a ‘smooth’ function of  $k$  in a neighbourhood<sup>5</sup> of  $k_F$  (for specification see further on) similar to that of non-interacting QPs, namely<sup>6</sup>  $\varepsilon_k^0 \propto k^2$ , with  $k := \|\mathbf{k}\|$ . In fact, for the special case of ‘Luttinger liquids’ (Haldane 1980, 1981, Luttinger 1963, Mattis and Lieb 1965, Dover 1968, Anderson 1997), without hereby specifying the dimension of the spatial space to which the corresponding system is confined, or marginal Fermi liquids (Varma, *et al.*, 1989, Littlewood and Varma 1991, Kotliar, *et al.*, 1991), we explicitly demonstrate that  $\varepsilon_k$  is *not* differentiable at  $k = k_F$  (see §§ IV.C and VI). As we shall emphasise later in this paper (§ IV.C), QPs in Fermi liquids are special in that their energy dispersion in the close vicinity of the Fermi surface is, apart from a scaling factor arising from the re-normalisation of the electron mass or the Fermi velocity, non-interacting-like. Our close inspection of the proof of the Luttinger (1961) theorem (§ IV), the contents of which we have spelled out in the Abstract, reveals that this ‘non-interacting-like’ assumption with regard to the dispersion of the QP energies is implicit in Luttinger’s (1961) proof, so that Luttinger’s theorem in essence amounts to a statement concerning the consistency of this assumption with the property  $\text{Im}\Sigma(k; \varepsilon) \sim \mp \alpha_k (\varepsilon - \varepsilon_F)^2$ ,  $\varepsilon \gtrless \varepsilon_F$ , for  $\varepsilon \rightarrow \varepsilon_F$ . In view of our rigorous demonstration with regard to non-differentiability of  $\varepsilon_k$  at  $k = k_F$  for Luttinger liquids, or marginal Fermi liquids, which invalidates Luttinger’s implicit assumption, we are left to conclude that in principle nothing precludes existence of a non-Fermi-liquid behaviour for *metallic* systems<sup>7</sup> in spatial dimensions

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<sup>3</sup> From dimensional considerations it follows that for a system with linear dimension  $L$ , the separations between the energy levels scale like  $L^{-2}$  (Landau and Lifshitz 1980, p. 14).

<sup>4</sup> At places in the present text, such as here, we refer to ‘QP energies’ even though QPs may not be well-defined. This is justified by the fact that even in these cases the equation for the ‘QP energies’ (i.e. Eq. (14) below) does have a solution — the ill-defined nature of QPs in these cases is associated with the fact that the single-particle eigenstates corresponding to these energies (see Eq. (10) below) do *not* describe particle-like excitations (for details see in particular § IV.C.).

<sup>5</sup> Throughout the present work by ‘neighbourhood’ we refer to an *open* non-vanishing interval (e.g. along the real axis) or domain (on the complex plane) which, however, may be arbitrarily small.

<sup>6</sup> We have  $\varepsilon_k^0 := \hbar^2 k^2 / [2m_e] \equiv \varepsilon_{k_F}^0 + (\hbar^2 k_F / m_e)(k - k_F) + (\hbar^2 / [2m_e])(k - k_F)^2$ . The important feature of this energy dispersion is that it is of the general form  $\varepsilon_k^0 = \varepsilon_{k_F}^0 + \varsigma(k - k_F) + o(k - k_F)$  for  $k \rightarrow k_F$  (see § IV.C), where  $\varsigma$  is a *finite* constant (explicitly, in the case at hand we have  $\varsigma \equiv \hbar^2 k_F / m_e$ ) and  $o(k - k_F)$  stands for a function that in comparison with  $(k - k_F)$  is vanishingly small as  $k \rightarrow k_F$ . In this respect the property  $\varepsilon_k^0 \propto k^2$  is to us of relatively minor relevance. We point out that the coefficient of the *linear* term in the QP energy dispersion (here  $\varsigma$ ), if such a coefficient at all exists, has the relevance that through it such attributes as the Fermi velocity  $v_F$  and effective mass  $m_e^*$  can be assigned to the corresponding QP (see § IV.C, specifically Eqs. (20) and (21)).

<sup>7</sup> Our emphasise throughout this work on the *metallic* nature of the systems under consideration corresponds to the fact that

larger than unity. In particular, contrary to the general belief, the break-down of the many-body perturbation theory (Luttinger 1961, Anderson 1988, 1989, Varma, *et al.* 1989, Anderson 1990a,b, 1991, Littlewood and Varma 1991, Kotliar, *et al.* 1991, Anderson 1992, 1993, 1997) for a metallic system and the nature of the low-lying single-particle excitations in this stand in *no* direct relationship.<sup>8</sup>

For clarity of discussions, we briefly specify the scheme by which we classify a *metallic* system as a Fermi liquid. This scheme, which to our best knowledge has not been utilised earlier in a similar context, has the advantage of being general and independent of the details of the metallic system to which it is applied. To this end, we first point out that the behaviour  $\text{Im}\Sigma(k; \varepsilon) \sim \mp \alpha_k(\varepsilon - \varepsilon_F)^2$ ,  $\varepsilon \gtrless \varepsilon_F$ , for  $\varepsilon \rightarrow \varepsilon_F$ , which is specific to Fermi liquids, leads to (see §§ IV.C, V and Appendix C)  $\text{Re}\Sigma(k, \varepsilon) - \Sigma(k, \varepsilon_F) \sim \beta_k(\varepsilon - \varepsilon_F)$  for  $\varepsilon \rightarrow \varepsilon_F$ , with  $\beta_k$  a constant to be specified in Appendix C. The distinguishing aspect of  $\Sigma(k, \varepsilon)$  whose real and imaginary parts have such asymptotic forms is as follows.

**Condition A:**  $\Sigma(k_F; \varepsilon)$  is a *continuously differentiable* function<sup>9</sup> of  $\varepsilon$  in a neighbourhood of  $\varepsilon_F$ .

This condition suffices to guarantee a finite discontinuity  $Z_{k_F}$  in the momentum-distribution function  $n(k)$  at  $k = k_F$  (§ V). A finite  $Z_{k_F}$  is generally considered to signify a Fermi liquid. If this merely evidences a Fermi liquid, then condition (A) would constitute the necessary *and* sufficient condition for a metallic system to be one (see § V). However, condition (A) can be shown (see §§ IV.C and V) not to be sufficient for the QP energy to possess a ‘well-defined’ dispersion in a neighbourhood of  $k = k_F$ , that is one which is expressible in terms of a continuously differentiable function of  $k$  in this neighbourhood. For the existence of such a dispersion, the following condition is also required to be satisfied.

**Condition B:**  $\Sigma(k; \varepsilon_F)$  is a *continuously differentiable* function of  $k$  in a neighbourhood of  $k = k_F$ .

We therefore choose to consider a *metallic* system as a Fermi liquid provided the corresponding  $\Sigma(k; \varepsilon)$  satisfies both condition (A) and condition (B); this specifies the notion of ‘smooth’ introduced above. According to this, which is physically the most sound definition for Fermi liquids,<sup>10</sup> a finite  $Z_{k_F}$ , although *necessary*, is not *sufficient* for a system to be a Fermi liquid. We explicitly demonstrate (§ VI) that whereas  $\Sigma(k; \varepsilon_F)$  corresponding to a Fermi liquid is (by definition) a continuously differentiable function of  $k$  in a neighbourhood of  $k = k_F$ , the same *may not* apply to  $\Sigma(k; \varepsilon)$  when  $\varepsilon \neq \varepsilon_F$ . We further demonstrate (§ VI) that  $\Sigma(k; \varepsilon)$  pertaining to a metallic system (whether Fermi liquid or otherwise), which is not continuously differentiable with respect to  $k$  in a neighbourhood of  $k = k_0$  (where  $k_0$  would possibly be  $k_F$ ) for *some*  $\varepsilon \neq \varepsilon_F$ , is *not* continuously differentiable with respect to  $k$  in a neighbourhood of  $k = k_0$  for *any*  $\varepsilon$ , with the possible exception of  $\varepsilon = \varepsilon_F$ ; this exceptional instance is (by definition) reserved for Fermi liquids (see above). To be explicit, consider  $\alpha_k$  and  $\beta_k$  which feature in the above expressions for the self-energy of a Fermi liquid (see also § V). Our above statements imply that these may *not* necessarily be continuously differentiable functions of  $k$  in a neighbourhood of  $k = k_F$ ; possible singularities associated with  $\partial\alpha_k/\partial k$  and  $\partial\beta_k/\partial k$ , as  $k \rightarrow k_F$ , are suppressed by  $(\varepsilon - \varepsilon_F)^2$  and  $(\varepsilon - \varepsilon_F)$  respectively for  $\varepsilon = \varepsilon_F$ . In § VI we establish relationships between the behaviour of  $\Sigma(k; \varepsilon)$  for  $k$  and  $\varepsilon$  in neighbourhoods of  $k_F$  and  $\varepsilon_F$  respectively and that of the momentum distribution function for  $k$  approaching  $k_F$ .

Aside from the above-indicated ‘smoothness’ assumption with regard to the dispersion of the QP energies which is implicit in Luttinger’s (1961) proof, there is a second aspect specific to this proof that makes the theorem inapplicable to metallic systems of particles interacting through the long-range Coulomb interaction function. Luttinger’s approach is based on the perturbation expansion of the self-energy operator in terms of the exact single-particle Green function (the terms in this expansion are represented by means of skeleton self-energy Feynman diagrams (Luttinger and Ward 1960)) and the *bare* electron-electron interaction function. It can be shown (§ IV.D) that a certain class of terms

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non-metallic systems are *by definition* non-Fermi liquids.

<sup>8</sup> We should like to emphasise that often from the contexts of arguments presented in the literature, in disapproval of the many-body perturbation theory, it is not evident what precisely is meant by the ‘perturbation theory’. In the present work we consider a series for the self-energy operator in terms of the *interacting* single-particle Green function, each term of which is diagrammatically represented by a *skeleton* diagram, as constituting a perturbation series. This we do entirely in the spirit of Luttinger’s (1961) work. We should point out that such a series is *not* a power series in the coupling constant of the electron-electron interaction and consequently does not conform to the orthodox definition for perturbation series.

<sup>9</sup> A function  $f(x)$  is *continuously differentiable* in the *closed* interval  $[a, b]$  provided firstly that it is continuous in  $[a, b]$  and secondly that its derivative exists at all points of the *open* interval  $(a, b)$  and coincides at all such points with a function which is continuous in  $[a, b]$ ;  $f(x)$  is continuously differentiable in a *neighbourhood* of  $x = x_0$  if  $x_0$  is *interior* to a finite interval, such as  $[a, b]$ , in which  $f(x)$  is continuously differentiable.

<sup>10</sup> The notion of a QP without an associated energy dispersion is devoid of physical significance.

in this expansion are unbounded as a consequence of non-integrable singularities corresponding to zero momentum-transfer scattering processes (infrared divergence). Therefore, for such systems a formal term-by-term analysis of the perturbation series for the self-energy operator can only be meaningful when the perturbation expansion is in terms of the dynamically-screened interaction function (Hubbard 1957)  $W(\varepsilon)$  (for a comprehensive discussion see Mattuck (1992, § 10.4)). In contrast to the *static* bare Coulomb interaction  $v_c$ ,  $W(\varepsilon)$  is a function with singularities along the real energy (i.e.  $\varepsilon$ ) axis (branch points, branch cuts, etc. — similar to the Green function and the self-energy operator) and a Lehmann-type spectral representation of this function reveals that these singularities coincide with energies of the neutral elementary excitations of the system (i.e. energies of excited  $N$ -electron states as measured with respect to the energy of the  $N$ -electron ground state). It follows that an analysis of the self-energy operator along the lines of Luttinger (1961) requires *additional* knowledge with regard to the dispersion of these neutral excitations. In other words, in this case the dispersion of the QP energies and that of neutral excitations cannot be considered separately, but must be dealt with in conjunction. This is of relevance particularly to uniform two-dimensional systems of electrons whose spectrum of coherent neutral excitations (i.e. plasmons) is gap-less, implying some non-negligible amount of interference amongst the neutral and the single-particle, i.e. QP, excitations.

Our present work has been motivated by a long-standing discussion in the literature with regard to the unusual properties of layered high- $T_c$  compounds in their normal states. On the one hand these unusual properties have been ascribed to the fact that the normal metallic states of these systems not Fermi-liquid states (Anderson 1988, 1989, Varma, *et al.*, 1989, Anderson 1990a,b, 1991, Littlewood and Varma 1991, Kotliar, *et al.*, 1991, Anderson 1992, 1993, 1997) while, on the other hand, according to the Luttinger (1961) theorem dealt with in the present work, metallic interacting systems in spatial dimensions larger than unity must be Fermi liquids. As emphasised by Luttinger (1961), the validity of the Luttinger theorem is conditional to that of the many-body perturbation theory for the self-energy operator to *all* orders. In this light, it is asserted that a non-Fermi-liquid metallic behaviour of a system would necessarily imply break-down of the many-body perturbation theory for this system (Anderson 1997). These observations give rise to the following questions.

(1) Can interacting electron systems in spatial dimensions  $d$  larger than unity, in particular for  $d = 2$ , be non-Fermi liquids?

(2) does the many-body perturbation theory break down in cases where the metallic systems under consideration are non-Fermi liquids?

Concerning question (1), the analyses presented in this work lead us to the conclusion that in principle *nothing* stands in the way of realisation of non-Fermi-liquid states in spatial dimensions  $d$  larger than unity. In fact the very strict conditions (A) and (B), introduced above, imposed on the self-energy of Fermi liquids, suggests that non-Fermi liquid metallic systems (in  $d > 1$ ) may not be as uncommon as generally perceived. We do not touch upon the above question (2) and refrain from statements that are likely to be speculative at this stage. In this connection we point out that, since the existence of non-Fermi liquids (in particular for  $d = 2$ ) and the validity of the many-body perturbation theory as applied to these are not *a priori* mutually exclusive, the question with regard to breakdown of this theory is not as urgent as it would be otherwise. This, however, does not diminish the relevance of question (2). We should like to emphasise that in calculating  $\Sigma(k; \varepsilon)$ , whether perturbatively or otherwise, account has to be taken of the fact that any approximation scheme that involves indiscriminate Taylor expansions of functions of energy and momentum (which expansions implicitly imply *continuous differentiability* of these functions in the pertinent regions of energy and momentum) may inhibit a non-differentiable  $\Sigma(k; \varepsilon)$  as function of both  $k$  and  $\varepsilon$ , and therefore a non-Fermi liquid behaviour, from being obtained.

The above questions, (1) and (2), have been subject of intensive study in recent years, the main body of the results thus far obtained pointing towards a Fermi-liquid state in two spatial dimensions. In spite of these, it has as yet *not* proved possible unequivocally to rule out the existence of non-Fermi-liquid states in two-dimensional metallic systems. This can be ascribed to two main reasons.

(i) Owing to the complexity of the many-body problem at hand, application of reliable theoretical techniques must of necessity be accompanied by simplifying approximations, the integrity of which may be a matter of dispute.

(ii) Even relevance of certain calculated quantities to the problem at hand has been matter of debate.

Both these aspects are aptly represented in the following: Engelbrecht and Randeria (1990) have found no evidence for the breakdown of the many-body perturbation theory and the Fermi-liquid theory in a dilute two-dimensional system of fermions interacting through a short-range repulsive potential, contradicting the suggestion made by Anderson (1990b). It appears, however, that the phase shift as calculated by Engelbrecht and Randeria is *not* that which is

encountered in the arguments by Anderson (see Engelbrecht and Randeria 1991, Anderson 1991).<sup>11</sup>

Work by Fujimoto (1990), Fukuyama, Nariakiyo and Hasegawa (1991) and Fukuyama, Hasegawa and Nariakiyo (1991) on the two-dimensional Hubbard model with repulsive on-site interaction  $U$  within the  $t$ -matrix approximation (which is appropriate to the low-density limit) has indicated that, whereas for  $k \neq k_F$ ,  $\text{Im}\Sigma(k, \varepsilon)$  retains the conventional Fermi-liquid form in three spatial dimensions (presented above),<sup>12</sup>  $\text{Im}\Sigma(k = k_F, \varepsilon) \sim (\varepsilon - \varepsilon_F)^2 \ln |\varepsilon - \varepsilon_F|$  as  $\varepsilon \rightarrow \varepsilon_F$ . This singular contribution, obtained earlier by Hodges, Smith and Wilkins (1971) and Bloom (1975) concerning two-dimensional Fermi systems, is not sufficiently strong to render the self-energy non-Fermi-liquid like.<sup>13</sup> Serene and Hess (1991), using the conserving ‘fluctuating-exchange (FLEX) approximation’ on a finite but large lattice, have also not found evidence for a non-Fermi-liquid behaviour in the two-dimensional Hubbard model. In a more recent work, Yokoyama and Fukuyama (1997) have arrived at the conclusion that in the two-dimensional Hubbard model with repulsive on-site interaction, the process of forward scattering gives rise to an anomalous contribution to the self-energy and consequently to a vanishing quasi-particle weight factor  $Z_{k_F}$ . This result, Yokoyama and Fukuyama (1997) asserted, would be “a microscopic demonstration of the claim by Anderson [(1990a,b), (1993)]”. We (Farid 1999a) have shown that the indicated anomalous contribution is *entirely* a consequence of the violation of a crucial symmetry in the momentum space (associated with the time-reversal symmetry of the ground state of the system under consideration) by the particle-particle correlation function employed by these workers.

Momentum-space perturbative renormalisation-group calculations by Shankar (1991, 1994) have also borne out a Fermi-liquid picture of fermions in two spatial dimensions. Shankar (1994) enumerated, however, a number of possibilities that in principle may render this finding, in its generality, invalid (see § XI in (Shankar 1994)). Work by Castellani, Di Castro (1994), Castellani, Di Castro and Metzner (1994), and Metzner, Castellani and Di Castro (1998) modelled on the treatment of one-dimensional interacting systems put forward by Dzyaloshinskii and Larkin (1973), which exploits the conservation laws and the associated Ward identities, and treats  $d$ , the spatial dimension, as a *real* variable, establishes that metallic systems with strong forward scattering (i.e. that corresponding to small momentum transfers) are Fermi liquids in  $d > 1$ , even though for  $d \neq 3$  some properties of these systems differ from those pertaining to conventional Fermi liquids in  $d = 3$ . Here the limited applicability of the formalism to systems with dominant forward scattering does not rule out non-Fermi-liquid systems in  $d > 1$ . In a forthcoming work (Farid 1999b) we present a detailed study concerning some limiting aspects associated with the technique employed in these studies.

Systems of particles interacting through long-range repulsive interaction functions  $v(\mathbf{r} - \mathbf{r}')$  which for large  $\|\mathbf{r} - \mathbf{r}'\|$  behave like  $\sim 1/\|\mathbf{r} - \mathbf{r}'\|^{2-d}$ , when  $1 < d < 2$ , and like  $\sim \ln \|\mathbf{r} - \mathbf{r}'\|$ , when  $d = 2$ , have been shown to resemble one-dimensional Luttinger liquids for  $1 < d < 2$  and a “ $Z_{k_F} = 0$  Fermi liquid” for  $d = 2$  (Bares and Wen 1993, Kwon, Houghton and Marston 1995). We note that in metals the Coulomb interaction function, which behaves like  $1/\|\mathbf{r} - \mathbf{r}'\|$ , is screened through the mediation of the charge polarisation fluctuations (Hubbard 1957); within the framework of the random-phase approximation (RPA), the screened interaction function in the *static* limit can be shown to behave like  $\sim \cos(2k_F\|\mathbf{r} - \mathbf{r}'\|)/\|\mathbf{r} - \mathbf{r}'\|^3$  (Fetter and Walecka 1971, pp. 178 and 179; Ashcroft and Mermin 1981, p. 343). It follows that the above long-range interactions should have their origin in processes that are not related to the electrostatic electron-electron interaction. The only known interaction that remains long-ranged in metals corresponds to that of electronic currents mediated by the exchange of transverse photons (Holstein, Norton, and Pincus 1973, Reizer 1989, 1991). Lack of the *static* screening of this interaction leads to the asymptotic behaviour of the self-energy  $\Sigma(k; \varepsilon_k)$  (on-the-mass-shell self-energy),<sup>14</sup> as  $k \rightarrow k_F$  (or  $\varepsilon_k \rightarrow \varepsilon_F$ ), to be that characteristic of the marginal Fermi liquids (Varma, *et al.*, 1989, Littlewood and Varma 1991, Kotliar, *et al.* 1991); on the other hand, for the fixed  $k \neq k_F$ ,  $\Sigma(k; \varepsilon)$  behaves Fermi-liquid like as  $\varepsilon \rightarrow \varepsilon_F$  (Holstein, Norton, and Pincus 1973). We note that the  $\Sigma(k; \varepsilon)$  considered here does *not* involve the effects of the Coulomb interaction beyond a mean-field level; it merely describes the self-energy of the current-current interaction whose bare coupling constant is proportional to the square of the ratio of the Fermi velocity to the light velocity in vacuum ( $\sim 10^{-4}$ ) and therefore is not of dominating influence except in extremely pure metals and at very low temperatures. We shall not enter into further details concerning this

<sup>11</sup> For a detailed discussion of the singular effective interaction amongst QPs, as encountered in the arguments by Anderson (1991), see Stamp (1993); see also Houghton, Kwon, and Marston (1994).

<sup>12</sup> Throughout,  $\sim$  indicates that the corresponding asymptotic relations are correct up to multiplicative constants.

<sup>13</sup> It can be shown (see Appendix C) that the associated  $\text{Re}\Sigma(k_F; \varepsilon) - \Sigma(k_F; \varepsilon_F) \sim (\varepsilon - \varepsilon_F)$  so that  $\Sigma(k_F; \varepsilon)$  is a continuously differentiable function of  $\varepsilon$  in a neighbourhood of  $\varepsilon = \varepsilon_F$  and thus, on account of condition (A) discussed above, gives rise to a finite  $Z_{k_F}$ .

<sup>14</sup> For the equation satisfied by the QP energy dispersion  $\varepsilon_k$  see Eq. (18) below.

subject here <sup>15</sup> and refer the reader to the cited literature. <sup>16</sup>

The organisation of this work is as follows. In § II we present the Lehmann spectral representation for the single-particle Green function and clarify certain elements that are of particular relevance to our analysis of the Luttinger (1961) theorem. In § III we rigorously define the notion of QP and derive equations from which QP energies and wavefunctions can be obtained. Here we indicate the necessary steps to be undertaken for obtaining complex-valued QP energies. We devote § IV to the main objective of our work, namely a detailed analysis of the Luttinger theorem. In § V we briefly review and discuss a theorem due to Migdal (1957) in light of our findings in § IV. In § VI we compare Fermi liquids with non-Fermi liquids in terms of the ‘smoothness’ properties of  $\Sigma(k; \varepsilon)$ , as a function of both  $k$  and  $\varepsilon$ . In § VII we conclude this work by a brief discussion and a review of our main results. In Appendix A we introduce and discuss a number of mathematical concepts that we repeatedly encounter in the present work. Here we further attempt to clarify the physical relevance of the introduced notions by means of some simple examples. In Appendix B we derive the leading asymptotic contribution to the self-energy operator  $\Sigma(\varepsilon)$  at large  $|\varepsilon|$ . For this, the next-to-the-leading-order asymptotic contribution to the single-particle Green function needs be calculated; in the same Appendix we present this contribution as well as some details underlying its derivation. In Appendices C and D we consider the next-to-the-leading-order asymptotic term to the self-energy pertaining to Fermi- and marginal Fermi-liquids (Appendix C) and the Luttinger liquid (Appendix D). Here we separately deal with the cases corresponding to  $k = k_F$  and  $k \neq k_F$ . In Appendix D we explicitly demonstrate that the self-energy of the one-dimensional Luttinger model is not continuously differentiable at the Fermi points, in full conformity with a general result established in Appendix C. *In this work we identify electrons with spin-less fermions.*

## II. THE SINGLE-PARTICLE GREEN FUNCTION

Here we explicitly deal with the single-particle Green function. In addition to an exposition of the formal significance of the singularities of this function, we discuss three specific and distinct energies  $\mu$ ,  $\mu_N$  and  $\mu_{N+1}$ , that are invariably (but un-justifiably) identified in the literature concerning the free-electron system.

Consider the following Lehmann (1954) representation (Fetter and Walecka 1971) for the (‘physical’) single-particle Green function in the coordinate-free representation

$$G(\varepsilon) = \hbar \sum_s \Lambda_s \Lambda_s^\dagger \left\{ \frac{\Theta(\mu - \varepsilon_s)}{\varepsilon - \varepsilon_s - i\eta} + \frac{\Theta(\varepsilon_s - \mu)}{\varepsilon - \varepsilon_s + i\eta} \right\}, \quad \eta \downarrow 0, \quad (1)$$

where

$$\Lambda_s := \begin{cases} \langle \Psi_{N-1,s} | \hat{\psi} | \Psi_{N,0} \rangle, & \text{when } \varepsilon_s < \mu, \\ \langle \Psi_{N,0} | \hat{\psi} | \Psi_{N+1,s} \rangle, & \text{when } \varepsilon_s > \mu \end{cases} \quad (2)$$

denotes a ‘Lehmann amplitude’, and

$$\varepsilon_s := \begin{cases} E_{N,0} - E_{N-1,s}, & \text{when } \varepsilon_s < \mu, \\ E_{N+1,s} - E_{N,0}, & \text{when } \varepsilon_s > \mu. \end{cases} \quad (3)$$

Above  $|\Psi_{M,s}\rangle$  stands for the *exact*  $M$ -particle eigenstate of the interacting system and  $E_{M,s}$  for the corresponding eigenenergy;  $s$  denotes the set of all quantum numbers that uniquely specify  $|\Psi_{M,s}\rangle$ , (we choose  $s = 0$  to signify the ground state which we assume to be non-degenerate);  $\hat{\psi}$  is the annihilation field operator. <sup>17</sup> Here the ‘chemical potential’  $\mu$  is *any* real value which satisfies  $\mu_N \leq \mu \leq \mu_{N+1}$ , where  $\mu_N := E_{N,0} - E_{N-1,0}$  and  $\mu_{N+1} := E_{N+1,0} - E_{N,0}$ . That such a  $\mu$  should exist follows from the requirement of stability of the ground state, meaning that  $\varepsilon_g := (E_{N+1,0} - E_{N,0}) - (E_{N,0} - E_{N-1,0}) \equiv \mu_{N+1} - \mu_N$  be *non-negative*. For a system with uniform electron distribution, which is

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<sup>15</sup> In particular we do not touch upon the subject of electrons interacting with a gauge field, such as is the case in two-dimensional electron systems exposed to external magnetic field (the gauge field here being the statistical Chern-Simons field), where *in principle* at fractional Landau-level filling factors  $\nu$  with even denominators (specifically at and close to  $\nu = 1/2$ ) the states are metallic but may be non-Fermi liquids (Kalmeyer and Zhang 1992, Halperin, Lee and Read 1993).

<sup>16</sup> For a concise review see (Tselik 1995, Ch. 12).

<sup>17</sup> Throughout this work,  $N$  denotes the actual number of the electrons in the system.

thus a system in the thermodynamic limit, it holds  $\mu_{N+1} = \mu_N + \mathcal{O}(N^{-p})$ , with  $p > 0$ .<sup>18</sup> Therefore with regard to uniform systems, we are considering a case in which  $(\mu_N, \mu_{N+1})$  is a *finite* infinitesimal open interval. In § IV.C we shall see that  $\mu_N \equiv \varepsilon_F$ , the Fermi energy of the system of  $N$  electrons.

It can be easily verified that (see Appendix A and in particular § A.3, for our notational conventions)

$$\tilde{G}(z) := \hbar \sum_s \frac{\Lambda_s \Lambda_s^\dagger}{z - \varepsilon_s} \quad (4)$$

is the analytic continuation (Whittaker and Watson 1927, pp. 96-98, Titchmarsh 1939, pp. 138-164) of the (‘physical’) single-particle Green function  $G(\varepsilon)$  into the complex  $z$ -plane, that it is  $\tilde{G}(z)$  on the physical RS (see Appendix A);  $G(\varepsilon)$  is obtained from  $\tilde{G}(z)$  according to the following prescription<sup>19 20</sup>

$$f(\varepsilon) := \lim_{\eta \downarrow 0} \tilde{f}(\varepsilon \pm i\eta), \quad \text{for } \varepsilon \gtrless \mu. \quad (5)$$

For a *finite* system, the Lehmann representation indicates that  $\tilde{G}(\varepsilon) \equiv G(\varepsilon)$  has *poles*<sup>21</sup> at  $\varepsilon = \varepsilon_s$  for all  $s$  (see Eq. (3)). In the thermodynamic limit, the singular points of  $\tilde{G}(z)$  correspond to branch points (isolated singularities are not *a priori* excluded) and continua of branch cuts covering (parts of) the *real* energy axis (see § A.4). For finite systems as well as those in the thermodynamic limit,  $\tilde{G}(z)$  is analytic for *all*  $z$  with  $\text{Im}(z) \neq 0$ .

In § III we shall encounter the inverse operator  $\tilde{G}^{-1}(z)$ . In view of this, we point out that from the representation in Eq. (4) it can be easily demonstrated that  $\tilde{G}(z)$  has *no* complex zeros (or, better, *zero eigenvalues*) on the  $z$ -plane (Luttinger 1961). Thus  $\tilde{G}^{-1}(z)$ , similar to  $\tilde{G}(z)$ , is analytic for *all*  $z$  with  $\text{Im}(z) \neq 0$ .

### III. QUASI-PARTICLES; THEIR “ENERGIES” AND WAVEFUNCTIONS

As we have mentioned in § I, to quasi-particles correspond one-particle wavefunctions which are eigenfunctions of some one-particle-like Schrödinger equation corresponding to an energy-dependent non-Hermitian one-body Hamiltonian, the potential-energy part of which consists of the self-energy operator<sup>22</sup> and of possibly a non-trivial contribution due to an external potential (such as an ionic potential in a solid). We devote this Section to derivation as well as to the discussion of the equations for QP energies and wavefunctions. Some specific properties of the equation for the QP energies are of considerable relevance to our discussion of the Luttinger (1961) theorem in § IV.

In the representation-free form and for the complex energy  $z$ , from the Dyson equation  $\tilde{G}(z) = \tilde{G}_0(z) + \tilde{G}_0(z)\tilde{\Sigma}(z)\tilde{G}(z)$ , with  $\tilde{G}_0(z)$  the single-particle Green function pertaining to the non-interacting system, one obtains

$$\tilde{G}^{-1}(z) = \frac{1}{\hbar} \{zI - \tilde{\mathcal{H}}(z)\} \quad (6)$$

<sup>18</sup> The value  $p = 1$  as given in, e.g., Fetter and Walecka (1971, p. 75), is incorrect.

<sup>19</sup> Below  $\tilde{f}(z)$  ( $f(\varepsilon)$ ) is a representative for any function of  $z$  ( $\varepsilon$ ) that we encounter in the present work.

<sup>20</sup> It should be noted that the prescription given in Eq. (5) is connected to our convention with regard to the time-Fourier transform of the time-dependent functions, namely  $f(\varepsilon) := \int_{-\infty}^{\infty} dt F(t) \exp(+i\varepsilon t/\hbar)$ . Had we chosen  $\exp(-i\varepsilon t/\hbar)$  rather than  $\exp(+i\varepsilon t/\hbar)$ , ‘ $\pm$ ’ would have to be ‘ $\mp$ ’.

<sup>21</sup> There is a subtlety involved here. For an open stable system, the spectrum consists of both a discrete and a continuous part. The former part corresponds to bound states, and the latter to scattering states; the continuous part of the spectrum gives rise to branch-cut discontinuities in the energy-dependent correlation functions of this system. Further, even though in such a system the number of particles is finite, the set of discrete one-particle excitation energies has an accumulation point which, as indicated in § A.1, is *not* an isolated singularity of  $\tilde{G}(z)$ . For closed and finite systems (that is those with impenetrable boundaries), the completeness of the one-particle eigenstates implies that the set of one-particle eigenenergies  $\{\varepsilon_s\}$  possesses at least one accumulation point (this follows from the Bolzano-Weierstrass theorem — see Whittaker and Watson (1927, pp. 12 and 13)). Therefore, the non-isolated nature of the singular points of, for example,  $\tilde{G}(z)$  is *not exclusively* a peculiarity of systems in the thermodynamic limit.

<sup>22</sup> In spite of the fact that according to our present convention  $\Sigma(k; \varepsilon)$  has dimension  $s^{-1}$ , i.e. inverse second, we refer to it as *self-energy*.

where  $I$  denotes the unit operator in the single-particle Hilbert space and

$$\tilde{\mathcal{H}}(z) := H_0 + \hbar \tilde{\Sigma}(z). \quad (7)$$

Any singular point of  $\tilde{G}(z)$  is a solution of  $\det(\tilde{G}^{-1}(z)) = 0$ , or, making use of Eq. (6), of

$$\det(zI - \tilde{\mathcal{H}}(z)) = 0. \quad (8)$$

This is reminiscent of the equation for the energies of the ‘ideal’ (i.e. non-interacting) QPs, namely  $\det(zI - H_0) = 0$ ; the difference between the two arises from the difference between  $\tilde{\mathcal{H}}(z)$  and  $H_0$  which according to Eq. (7) is equal to  $\hbar \tilde{\Sigma}(z)$ . The latter, a function of the complex energy parameter  $z$ , is a non-Hermitian operator; even for  $z \rightarrow \varepsilon \pm i\eta$ ,  $\eta \downarrow 0$ , for  $\varepsilon \gtrless \varepsilon_F$ , leading to  $\tilde{\Sigma}(z) \rightarrow \Sigma(\varepsilon)$  (see Eq. (5) above),  $\tilde{\mathcal{H}}(z) \equiv \mathcal{H}(\varepsilon)$  is *not* in general Hermitian. This implies, among others, that, unless  $\tilde{\mathcal{H}}(z)$  is Hermitian, the spectral decomposition of  $\tilde{\mathcal{H}}(z)$  involves the *distinct* sets of left and right eigenvectors,  $\{\tilde{\phi}_\ell(z)\}$  and  $\{\tilde{\psi}_\ell(z)\}$  respectively. Since the sets of eigenvalues pertaining to the latter two sets are up to ordering identical (Morse and Feshbach 1953, p. 885), by choosing the appropriate ordering, the spectral (or *bi-orthonormal* (Morse and Feshbach 1953, pp. 884-886, Layzer 1963) — see later) representation of  $\tilde{\mathcal{H}}(z)$  is as follows

$$\tilde{\mathcal{H}}(z) = \sum_{\ell} \tilde{E}_\ell(z) \tilde{\psi}_\ell(z) \tilde{\phi}_\ell^\dagger(z), \quad (9)$$

where  $\tilde{E}_\ell(z)$  denotes the common eigenvalue corresponding to  $\tilde{\phi}_\ell(z)$  and  $\tilde{\psi}_\ell(z)$ :

$$\tilde{\phi}_\ell^\dagger(z) \tilde{\mathcal{H}}(z) = \tilde{E}_\ell(z) \tilde{\phi}_\ell^\dagger(z); \quad \tilde{\mathcal{H}}(z) \tilde{\psi}_\ell(z) = \tilde{E}_\ell(z) \tilde{\psi}_\ell(z). \quad (10)$$

For  $\tilde{E}_\ell(z) \neq \tilde{E}_{\ell'}(z)$  it holds  $\langle \tilde{\phi}_\ell(z), \tilde{\psi}_{\ell'}(z) \rangle = \delta_{\ell, \ell'}$  (Morse and Feshbach 1953, p. 885). In the case of degeneracy, that is  $\tilde{E}_\ell(z) = \tilde{E}_{\ell'}(z)$  for  $\ell \neq \ell'$ , the degenerate left and right eigenvectors can be made orthogonal through a Gram-Schmidt orthogonalisation procedure. When the left and right eigenfunctions are simultaneously bases of the unitary irreducible representations of the symmetry group of the QP Schrödinger equation, Eq. (10), the degenerate eigenvectors are automatically orthogonal as long as they belong to different unitary irreducible representations (Cornwell 1984, pp. 81-83).<sup>23</sup> Further,  $\{\tilde{\phi}_\ell(z)\}$  and  $\{\tilde{\psi}_\ell(z)\}$  satisfy the completeness relation (Morse and Feshbach 1953, p. 886)<sup>24</sup>

$$\sum_{\ell} \tilde{\psi}_\ell(z) \tilde{\phi}_\ell^\dagger(z) = I, \quad (11)$$

or  $\sum_{\ell} \tilde{\psi}_\ell(\mathbf{r}; z) \tilde{\phi}_\ell^*(\mathbf{r}'; z) = \delta(\mathbf{r} - \mathbf{r}')$ .

Because  $\tilde{\mathcal{H}}^\dagger(z) = \tilde{\mathcal{H}}(z^*)$ , for  $\text{Im}(z) \neq 0$ , a property that follows from  $\tilde{\Sigma}^\dagger(z) = \tilde{\Sigma}(z^*)$  (Appendix B in DuBois (1959a), Luttinger 1961), it can be readily shown that, for  $\text{Im}(z) \neq 0$ ,

$$\tilde{\phi}_\ell(z) \equiv \tilde{\psi}_\ell(z^*), \quad \tilde{\psi}_\ell(z) \equiv \tilde{\phi}_\ell(z^*), \quad \tilde{E}_\ell(z^*) = \tilde{E}_\ell^*(z). \quad (12)$$

As we shall see later, the last result<sup>25</sup> is of particular significance to our considerations. By defining  $E_\ell(\varepsilon)$  in accordance with Eq. (5), for  $\text{Im} E_\ell(\varepsilon)$  the following must hold

<sup>23</sup> Consider, for instance, the uniform-electron system with which we deal in detail in the present work. The symmetry group of this system is the continuous translation group (which is Abelian) and therefore the corresponding bases for the (one-dimensional) unitary irreducible representations are specified by wave-vectors  $\mathbf{k}$ . The time-reversal symmetry of the problem implies degeneracy of eigenfunctions at  $\mathbf{k}$  and  $-\mathbf{k}$  (“Kramers’ degeneracy” (Landau and Lifshitz 1977, pp. 223-226)). Since for  $\mathbf{k} \neq \mathbf{0}$ ,  $\mathbf{k} \neq -\mathbf{k}$ , the eigenfunctions corresponding to  $\mathbf{k}$  and  $-\mathbf{k}$  are therefore automatically orthogonal. Note that, since our ‘electrons’ are spin-less, for the application of the Kramers theorem we *cannot* rely on the condition that the total spin of the system is half-integer. However, we can rely on the fact that *all* irreducible representations of the translation group corresponding to  $\mathbf{k} \neq \mathbf{0}$  are *essentially* complex.

<sup>24</sup> The left-hand side of this relation is the so-called *idemfactor* of the one-particle Hilbert space.

<sup>25</sup> This result is the expression of the Riemann-Schwarz reflection principle (Titchmarsh 1939, p. 155) which follows from the analyticity of  $\tilde{\Sigma}(z)$  over the entire complex  $z$ -plane, with the exception of the real  $\varepsilon$ -axis, and the fact that over the finite (even though infinitesimal) *open* interval  $(\mu_N, \mu_{N+1})$ ,  $\Sigma(\varepsilon)$ , or  $G(\varepsilon)$ , is real valued.



$$\text{Im}E_\ell(\varepsilon) \begin{matrix} \leq \\ \geq \end{matrix} 0, \text{ for } \varepsilon \begin{matrix} > \\ < \end{matrix} \mu. \quad (13)$$

Violation of these inequalities signifies breakdown of causality, or instability of the ground state due to its collapse into a lower-energy state; to appreciate this, note that (combine Eqs. (6) and (11))  $\tilde{G}(z) = \hbar \sum_\ell \tilde{\psi}_\ell(z) \tilde{\phi}_\ell^\dagger(z) / (z - \tilde{E}_\ell(z))$  and compare this with the Lehmann representation in Eq. (1) (see also Eq. (3)).

From the spectral representation for  $\tilde{\mathcal{H}}(z)$  in Eq. (9) above, together with Eq. (11), the equivalence of Eq. (8) with the set of equations

$$\tilde{E}_\ell(z) = z, \quad \forall \ell, \quad (14)$$

is readily established. As we have mentioned in § II, singular points of  $G(\varepsilon)$ , according to the Lehmann representation, coincide with the real-valued quantities  $\varepsilon_s$  (see Eq. (3)). Since Eq. (14) is the equation for these singular points, it may be expected that by taking the appropriate limits  $z \rightarrow \varepsilon \pm i\eta$ , with  $\eta \downarrow 0$  (see Eq. (5)), Eq. (14) should transform into an equation with real-valued solutions  $\varepsilon_s$ ; owing to the analyticity of  $\tilde{G}^{-1}(z)$  for  $\text{Im}(z) \neq 0$  (see § II), it follows that Eq. (14) cannot be satisfied for any  $z$  with  $\text{Im}(z) \neq 0$  (for a physical interpretation of this statement see the following paragraph). As we shall demonstrate below, in the thermodynamic limit the number of real-valued solutions of Eq. (14), with  $z \rightarrow \varepsilon \pm i\eta$  (see Eq. (5)), is limited. Further, in the same limit, these real-valued solutions are in general not poles, in contrast with what the Lehmann representation in Eq. (1) (or Eq. (4)) may suggest. This is not a contradiction, since as we indicate in § A.4, analytic properties of correlation functions in the thermodynamic limit are fundamentally different from those of finite systems.

It is often mentioned that finite lifetimes of QPs is reflected in the energies of these being complex valued. This statement, without further qualification, is misleading. To clarify this, consider Eq. (14) and suppose that it were satisfied by  $z = z_0$  with  $\text{Im}(z_0) \neq 0$ . Owing to the reflection property of  $\tilde{E}_\ell(z)$  presented in Eq. (12), it follows that  $z = z_0^*$  must also satisfy Eq. (14). This *cannot* be the case, since depending on whether  $\text{Re}(z_0)$  is less than or larger than  $\mu$ , one of the two solutions  $z_0$  and  $z_0^*$  signals violation of the principle of causality. In this connection, recall that the single-particle Green function has been defined in terms of the *time-ordered* product of two field operators in the Heisenberg representation (Fetter and Walecka 1971). Consequently, no  $z_0$  can satisfy Eq. (14) unless  $\text{Im}(z_0) = 0$ . Thus for a given  $\ell$ , Eq. (14) has either a real-valued solution or no solution at all; the possible real-valued solutions (see § A.1) are, in general, *not* isolated. To obtain complex-valued quasi-particle energies, the appropriate equation to be solved is the following

$$\tilde{\tilde{E}}_\ell(z) = z, \quad \forall \ell, \quad (15)$$

where  $\tilde{\tilde{E}}_\ell(z)$  represent *the* unique analytic function of which  $\tilde{E}_\ell(z)$  is one specific branch (see § A.3). The analytic function  $\tilde{\tilde{E}}_\ell(z)$  (i.e. any of its branches; there may be infinity of these, depending on the nature of the branch points of  $\tilde{E}_\ell(z)$ ) can in principle be calculated from the knowledge of  $\tilde{E}_\ell(z)$ . To this end, one starts from  $\tilde{E}_\ell(z)$  and analytically continues it across its branch cuts into a RS ‘above’ or ‘below’ the physical RS (see examples in §§ A.3 and A.4). This process can be repeated on any of the RSs and in this way one recovers all the possible branches of  $\tilde{\tilde{E}}_\ell(z)$ . From the specification presented in Eq. (5), it is readily observed that, to obtain the physically most relevant complex-valued solutions of Eq. (15), one has to analytically continue  $\tilde{E}_\ell(z)$  from the first (third) quadrant of the physical RS into the fourth (second) quadrant of a non-physical RS when  $\text{Re}(z) > \mu$  ( $\text{Re}(z) < \mu$ ). We note that solution  $z_0$  of  $\tilde{\tilde{E}}_\ell(z) = z$  with  $\text{Re}(z_0) > \mu$  corresponds to a QP that propagates and attenuates forwards in time (i.e. it concerns a ‘particle-like’ QP) and that with  $\text{Re}(z_0) < \mu$  to one that propagates and attenuates backwards in time (a ‘hole-like’ QP).

#### IV. DISCUSSION OF THE LUTTINGER THEOREM

We are now in a position to deal with the central objective of the present work. In doing so we distinguish three major steps in the proof by Luttinger (1961). We shall separately describe each of these steps and, when necessary, comment on the details. We conclude that Luttinger’s theorem does *not* exclude the existence of non-Fermi-liquid metallic states for systems in spatial dimensions larger than unity. Rather, it merely establishes that Landau’s Fermi-liquid theory for metals is consistent with the principles of the many-body theory of interacting systems.

## A. Preliminaries

Before proceeding with the proof of the Luttinger (1961) theorem, several remarks are in order.

First, our considerations in § II have made evident that  $\mu_N$  and  $\mu_{N+1}$  are two branch points of  $\tilde{G}(z)$  and  $\tilde{\Sigma}(z)$  and consequently of  $\tilde{E}_\ell(z)$ , for all  $\ell$ . For simplicity of notation, let  $\tilde{f}(z)$  denote any of these functions of  $z$ . The fact that  $\mu_N$  and  $\mu_{N+1}$  are branch points of  $\tilde{f}(z)$  implies that such expansion as  $\text{Im}\Sigma(k; \varepsilon) \sim \mp \alpha_k(\varepsilon - \varepsilon_F)^2$ ,  $\varepsilon \gtrless \varepsilon_F$ , cannot correspond to the leading-order term in a Taylor expansion of  $\tilde{\Sigma}(k; z)$  when  $z = \varepsilon_F$  is identified with either  $\mu_N$  or  $\mu_{N+1}$ . Rather, it corresponds to the leading-order term in an asymptotic expansion (Whittaker and Watson 1927, Ch. VIII, Copson 1965, Dingle 1973, Lauwerier 1977) of  $\text{Im}\Sigma(k; \varepsilon)$  for  $\varepsilon \rightarrow \varepsilon_F$  (see § A.2).

In view of the fact that for the uniform-electron system under consideration, the interval  $(\mu_N, \mu_{N+1})$  is infinitesimally small, it is tempting to identify  $\varepsilon_F$ ,  $\mu_N$  and  $\mu_{N+1}$ , as is commonly done in the literature. Such an identification is not justified. Our discussion of a Migdal's (1957) theorem in § V further clarifies this statement. We note that, strictly speaking,  $\varepsilon_F$  is identical with  $\mu_N$ . Thus the precise statement of the Luttinger theorem under consideration is as follows.<sup>26</sup>  $\text{Im}\Sigma(k; \varepsilon) \equiv 0$  for  $\varepsilon \in (\mu_N, \mu_{N+1})$ ;  $\text{Im}\tilde{\Sigma}(k; \varepsilon + i\eta) \equiv \text{Im}\Sigma(k; \varepsilon) \sim -\alpha_k(\varepsilon - \mu_{N+1})^2$  for  $\varepsilon \geq \mu_{N+1}$  and  $\text{Im}\tilde{\Sigma}(k; \varepsilon - i\eta) \equiv \text{Im}\Sigma(k; \varepsilon) \sim \alpha_k(\varepsilon - \mu_N)^2$  for  $\varepsilon \leq \mu_N$ .

Second, the above asymptotic expansions are not *uniform* (see § A.2), as is evident from the sign of  $\text{Im}\Sigma(k; z)$  which depends on the sign of  $\text{Im}(z)$  (or on  $\arg(z)$ ). Thus  $z = \mu_N$  and  $z = \mu_{N+1}$  are indeed branch points.

Third, analysis of the next-to-leading order terms in the asymptotic expansions of  $\tilde{\Sigma}(k; z)$  around  $z = \mu_N$  and  $z = \mu_{N+1}$  reveal that these terms involve the logarithm function, implying that both  $\mu_N$  and  $\mu_{N+1}$  are branch points of *infinite* order, that is to say  $\tilde{f}(z)$  (see § A.3) consists of infinitely many branches. Already the fact that  $\mu_N$  and  $\mu_{N+1}$  are not regular singularities of  $\tilde{f}(z)$  suggests that care should be exercised in defining the QP weights, in particular  $Z_{k_F}$  which features in the Migdal (1957) theorem and which determines the value of the discontinuity in the momentum distribution functions at the Fermi momentum  $\hbar k_F$ . We shall return to this point in § V.

In spite of our first remark above, for conformity with the existing texts in what follows we shall use the conventional notation (such as that in our Abstract) and proceed with the demonstration of the implicit assumption involved in the proof of the Luttinger (1961) theorem. To this end, we first summarise the main aspects of Luttinger's (1961) proof. This proof is based on the assumption of validity of the many-body perturbation theory and further rests on the analyses of the terms in an *implicit* perturbation expansion of the self-energy operator; the *implicit* nature of this expansion is associated with the fact that contributions in this are in terms of the single-particle Green function pertaining to the *interacting* system which is, in turn, a functional of the self-energy operator itself. Diagrammatically, the terms in this perturbation series are represented by the proper *skeleton* self-energy diagrams (Luttinger and Ward 1960), that is those connected self-energy diagrams that do not accommodate any simpler self-energy parts in them.

In the proof by Luttinger (1961), the skeleton diagrams are in terms of the isotropic bare electron-electron interaction function  $v(\mathbf{r} - \mathbf{r}')$  and this, as we shall discuss in § IV.D, implies that Luttinger's approach is *not* directly applicable to cases where  $v(\mathbf{r} - \mathbf{r}')$  is the long-range Coulomb interaction function.

Since the many-body perturbation theory underlies the proof by Luttinger (1961), the Luttinger theorem is valid insofar as this perturbation theory is valid. This aspect leads to the natural conclusion that metallic non-Fermi-liquid systems in spatial dimensions  $d$  larger than unity (see footnote 30 further on) fall outside the domain of applicability of the many-body perturbation theory, or that this theory should break down when applied to such systems. As has been emphasised earlier (Anderson 1993), validity or otherwise of a many-body perturbation series cannot be decided solely on the basis of its convergence or divergence, respectively. Despite convergence, the calculated limit may be unphysical.<sup>27</sup>

<sup>26</sup> In the work by Luttinger (1961), there is no explicit mention of the value or range of values of  $k$  for which  $\text{Im}\Sigma(k; \varepsilon) \sim \mp \alpha_k(\varepsilon - \varepsilon_F)^2$ ,  $\varepsilon \gtrless \varepsilon_F$ , is valid. This is, however, implicit in Eqs. (59)-(61) of Luttinger's work: Eq. (61) implies  $u$  to be small in magnitude ( $u := (\mu - x)$ ; in our notation,  $u := (\varepsilon_F - \varepsilon)$ ), so that by Eq. (60) one deduces  $t_1$ ,  $t_2$  and  $t_3$  to be small, requiring by Eq. (59) that  $k_1, k_2, k_3 \approx k_F$ . From  $\mathbf{k}_3 = \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}$ , using a geometrical construction, it is easily deduced that  $0 \leq k \lesssim 3k_F$ .

<sup>27</sup> The following example due to Simon (1970) should be clarifying: Consider the hydrogen-like Hamiltonian in three dimensions (here we use the Hartree atomic units):  $\mathcal{H} := -\nabla^2/2 - \lambda/r$ ,  $r \geq 0$ . The discrete energies of this Hamiltonian for  $\lambda > 0$  are  $\mathcal{E}_n(\lambda) = -\lambda^2/[2n^2]$ ,  $n = 1, 2, \dots$ . For  $\lambda \leq 0$  there are no discrete levels (no bound states). However,  $\mathcal{E}_n(\lambda)$  being a finite-order polynomial (technically, an *entire* function) of  $\lambda$ , the Rayleigh-Schrödinger perturbation expansion for, say, the ground-state energy (i.e.  $\mathcal{E}_1(\lambda)$ ), with  $-\lambda/r$  playing the role of the 'perturbation', yields *exactly*  $-\lambda^2/2$ , irrespective of whether  $\lambda > 0$  or

## B. Explicit analysis of the Luttinger theorem

After indicating that the perturbation contributions in the above-indicated expansion for  $\Sigma(k; \varepsilon)$  in terms of the skeleton self-energy diagrams below second order in the electron-electron interaction  $v$  are  $\varepsilon$ -independent, Luttinger (1961) considered in detail the second-order self-energy diagram that we have depicted in Fig. 1 (for facilitating direct comparison with the results in the work by Luttinger, below we adopt the notation that he employed as closely as possible). In doing so, Luttinger first evaluated the imaginary part of the self-energy operator corresponding to this diagram, with the single-particle Green function pertaining to the *interacting* system, that is  $G(\varepsilon)$  (represented by full line in Fig. 1), replaced by that pertaining to the *non-interacting* system, i.e.  $G_0(\varepsilon)$ .<sup>28</sup> From the well-known rules for transcribing Feynman diagrams into mathematical expressions in the momentum representation (Fetter and Walecka 1971, pp. 100-105), it is readily seen that the expression corresponding to the diagram in Fig. 1 involves three wave-vector integrations, over  $\mathbf{k}_1$ ,  $\mathbf{k}_2$  and  $\mathbf{k}_3$ . Following the fact that for the matrix elements of the electron-electron interaction,  $(\mathbf{k}, \mathbf{k}_3 | v | \mathbf{k}_1, \mathbf{k}_2) = \bar{v}(\mathbf{k} - \mathbf{k}_1) \delta_{\mathbf{k} - \mathbf{k}_1, \mathbf{k}_2 - \mathbf{k}_3}$  holds, where  $\mathbf{k}$  stands for the *external* wave-vector (i.e. the wave-vector associated with  $k := \|\mathbf{k}\|$  in the argument of  $\Sigma(k; \varepsilon)$ ), it would be tempting to eliminate one of the wave-vector integrals. This elimination would make the subsequent algebra extremely cumbersome. Rather, Luttinger (1961) transforms the three wave-vector integrals into three one-dimensional integrals over energies  $\varepsilon_{k_1}^0$ ,  $\varepsilon_{k_2}^0$  and  $\varepsilon_{k_3}^0$  of the non-interacting electrons, where

$$\varepsilon_k^0 := \frac{\hbar^2}{2m_e} k^2, \quad (16)$$

with  $m_e$  the bare-electron mass.<sup>29</sup>

A most crucial ingredient in Luttinger's proof consists of the recognition that, although the momentum conservation, enforced by the above  $\delta_{\mathbf{k} - \mathbf{k}_1, \mathbf{k}_2 - \mathbf{k}_3}$ , uniquely determines one of the three wave-vectors in terms of the other two, say  $\mathbf{k}_3$  in terms of  $\mathbf{k}_1$  and  $\mathbf{k}_2$ , nonetheless, barring one-dimensional systems (Luttinger 1961, footnote 5),  $\varepsilon_{k_1}^0$ ,  $\varepsilon_{k_2}^0$  and  $\varepsilon_{k_3}^0$  can be independently and continuously varied over some *finite* range of values.<sup>30</sup> This can be seen as follows: using the above dispersion for  $\varepsilon_k^0$  as well as  $\mathbf{k}_3 = \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}$ , it is readily seen that

$$\varepsilon_{k_3}^0 = \varepsilon_k^0 + \varepsilon_{k_1}^0 + \varepsilon_{k_2}^0 + \frac{\hbar^2}{m_e} \{ \mathbf{k}_1 \cdot \mathbf{k}_2 - \mathbf{k} \cdot \mathbf{k}_1 - \mathbf{k} \cdot \mathbf{k}_2 \}. \quad (17)$$

For fixed values of  $\varepsilon_{k_1}^0$  and  $\varepsilon_{k_2}^0$ , and in spite of  $\mathbf{k}_3 = \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}$ , for systems in spatial dimensions  $d > 1$ , the *orientational* freedom of vectors  $\mathbf{k}_1$  and  $\mathbf{k}_2$  with respect to each other as well as  $\mathbf{k}$  allows for *continuous* finite variations in  $\varepsilon_{k_3}^0$ . A crucial element in our arguments that follow, is derived from the fact that freedom in the independent variations of  $\varepsilon_{k_3}^0$  is a consequence of the expression in Eq. (17), which, in turn, follows from the *specific* form of the energy dispersion pertaining to *non-interacting* electrons in Eq. (16). As a matter of course the latter form is *not* unique in that one can construct energy dispersions  $\varepsilon_k^0$  different from that in Eq. (16) that equally give rise to the possibility of independent and continuous variations of  $\varepsilon_{k_j}^0$ ,  $j = 1, 2, 3$ , over some non-vanishing region. At the same time, it is

$\lambda \leq 0$  and independent of the magnitude of  $|\lambda|$ ; the coefficients of  $\lambda^m$ , for  $m = 0, 1$  and  $m > 2$ , in this expansion are identically vanishing. Evidently, the convergence of this perturbation series for  $\lambda \leq 0$  is a 'bogus' convergence.

<sup>28</sup> Explicitly, we have  $G_0(k; \varepsilon) \equiv \hbar \{ \Theta(k_F - k) / (\varepsilon - \varepsilon_k^0 - i\eta) + \Theta(k - k_F) / (\varepsilon - \varepsilon_k^0 + i\eta) \}$ ,  $\eta \downarrow 0$ ;  $\varepsilon_k^0$  is defined in Eq. (16).

<sup>29</sup> The approach by Luttinger (1961) aims at obtaining the leading-order term in  $(\varepsilon - \varepsilon_F)$  (in Luttinger's notation,  $u$ ), so that other integrals which merely determine numerical pre-factors are not explicitly dealt with by him. For clarity, consider a three-dimensional non-interacting system with  $\varepsilon_k^0$  the one-particle energy dispersion presented in Eq. (16). In the spherical-polar coordinate system  $(k, \phi, \theta)$  we have  $d^3k = k^2 \sin(\theta) dk d\phi d\theta \equiv \frac{1}{2} (2m_e/\hbar^2)^{3/2} \sqrt{\varepsilon_k^0} \sin(\theta) d\varepsilon_k^0 d\phi d\theta$ . In the treatment by Luttinger (1961),  $d^3k \propto d\varepsilon_k^0$  plays an all-important role.

<sup>30</sup> This is the instance where one-dimensional interacting systems are singled out as being in some fundamental way different from higher-dimensional interacting systems. The Luttinger model, which turns out to be akin to a large class of one-dimensional models (Haldane 1980, 1981), is however introduced in 1963 by Luttinger (1963) and was first correctly treated by Mattis and Lieb (1965). An earlier model to which the Luttinger model is similar, is due to Tomonaga (1950). For three different treatments of the Tomonaga-Luttinger models see Dzyaloshinskiĭ and Larkin (1973), Luther and Peschel (1974), and Everts and Schulz (1974). For reviews concerning one-dimensional systems of interacting fermions see Sólyom (1979), Mahan (1981, § 4.4), Voit (1994), Schönhammer (1997) and Schulz, Cuniberti and Pieri (1998).

not difficult to put forward energy dispersions that do *not* allow for such an independent continuous variation over a finite domain. We shall return to this point in § IV.C.

It is from the observation with regard to the possibility of independent variations of the three energies  $\varepsilon_{k_j}^0$ ,  $j = 1, 2, 3$ , that Luttinger (1961) has established the leading-order contribution to  $\text{Im}\Sigma(k; \varepsilon)$  of the second-order diagram in Fig. 1 to be proportional to  $(\varepsilon - \varepsilon_F^0)^2$  as  $\varepsilon \rightarrow \varepsilon_F^0$ .<sup>31</sup>

The second step and, from the viewpoint of our present work, the most crucial step in Luttinger's (1961) proof consists in establishing that, upon employing  $G(\varepsilon)$  rather than the  $G_0(\varepsilon)$  of the first step, the leading-order contribution to  $\text{Im}\Sigma(k; \varepsilon)$  pertaining to the second-order self-energy diagram in Fig. 1 is also proportional to  $(\varepsilon - \varepsilon_F)^2$ , for  $\varepsilon \rightarrow \varepsilon_F$ . This conclusion is of fundamental importance to the third part in Luttinger's proof, where, after having established the latter property, namely that the leading-order contribution to  $\text{Im}\Sigma(k; \varepsilon)$ , as  $\varepsilon \rightarrow \varepsilon_F$ , is unaffected by evaluating the contributions of the *skeleton* self-energy diagrams in terms of  $G_0(\varepsilon)$  rather than  $G(\varepsilon)$ , Luttinger demonstrated that to *all* orders in the perturbation theory  $\text{Im}\Sigma(k; \varepsilon) \sim (\varepsilon - \varepsilon_F)^2$  or explicitly, that any skeleton self-energy diagram (in terms of  $G_0(\varepsilon)$ ) of second and higher order in the *bare* electron-electron interaction  $v$  has a contribution to  $\text{Im}\Sigma(k; \varepsilon)$  proportional to  $(\varepsilon - \varepsilon_F^0)^{2m}$ , where  $m \geq 1$ , for  $\varepsilon \rightarrow \varepsilon_F^0$ .<sup>32</sup>

In this second part of his proof, Luttinger (1961) proceeded as follows. On evaluating the contribution of the diagram in Fig. 1 to  $\text{Im}\Sigma(k; \varepsilon)$  in terms of  $G(\varepsilon)$ , he made a simplifying approximation, which, roughly speaking, corresponds to replacing a Lorentzian by a Dirac  $\delta$ -function.<sup>33</sup> Upon this, Luttinger arrived at an expression which has the same formal structure as the one discussed above. Similar to the second-order expression for  $\text{Im}\Sigma(k; \varepsilon)$  in terms of  $G_0$ , the simplified expression in terms of  $G$  involves three wave-vector integrals over  $\mathbf{k}_1$ ,  $\mathbf{k}_2$  and  $\mathbf{k}_3$ . Now, at this place, Luttinger took a step which, as will become evident below, implicitly contains the property to be proven. Here Luttinger transforms the three wave-vector integrals into three energy integrals, the energy dispersions being  $\varepsilon_{k_1}$ ,  $\varepsilon_{k_2}$  and  $\varepsilon_{k_3}$ , to be compared with  $\varepsilon_{k_1}^0$ ,  $\varepsilon_{k_2}^0$  and  $\varepsilon_{k_3}^0$  considered above (see § IV.C). Note that the dispersions  $\varepsilon_{k_j}$ ,  $j = 1, 2, 3$ , correspond to the interacting (i.e. true) QPs and satisfy Eq. (14) or Eq. (18) below.<sup>34</sup> According to Luttinger, *all* the necessary mathematical steps that are to be taken in order to obtain, from the last-indicated integral, the leading-order contribution to  $\text{Im}\Sigma(k; \varepsilon)$ , as  $\varepsilon \rightarrow \varepsilon_F$ , are identical with those taken in dealing with  $\text{Im}\Sigma(k; \varepsilon)$  evaluated in terms of  $G_0$ . *This is an unjustified statement.* The dependence on  $k$  of  $\varepsilon_k$  being unknown, there is no *a priori* reason why  $\varepsilon_{k_3}$  can be varied continuously and independently of  $\varepsilon_{k_1}$  and  $\varepsilon_{k_2}$  over a finite range.<sup>35</sup> If the effect of electron-electron interaction on the dispersion of the QP energies close to the Fermi surface were only to re-normalise the QP mass, then of course for  $k$  in a neighbourhood of  $k = k_F$  one invariably had  $\varepsilon_k \sim \hbar^2 k^2 / (2m_e^*)$ , with  $m_e^*$  an electron's renormalised mass (Pines and Nozières 1966, Abrikosov, Gorkov and Dzyaloshinski 1975). In such an event, for  $k$  in a neighbourhood of  $k = k_F$ , Eq. (16) would hold for  $\varepsilon_k$  with  $m_e$  replaced by  $m_e^*$ , and thus the just-mentioned assertion by Luttinger (1961) were correct. However, as we shall demonstrate in § IV.C,  $\varepsilon_k \sim \hbar^2 k^2 / (2m_e^*)$  can only apply if  $\Sigma(k; \varepsilon)$  is of the Fermi-liquid type, thus establishing that Luttinger's proof amounts to a demonstration of consistency of the Fermi-liquid state with the many-body theory of interacting metals. This demonstration does not

<sup>31</sup> We note in passing that 'u<sup>2</sup>' in Eq. (60) of Luttinger (1960) is in error; the correct pre-factor is unity.

<sup>32</sup> It is known (Hodges, Smith and Wilkins 1971, Bloom 1975, Fujimoto 1990, Fukuyama, Narikiyo and Hasegawa 1991, Fukuyama, Hasegawa and Narikiyo 1991) that, in isotropic two-dimensional interacting systems,  $\text{Im}\Sigma(k = k_F; \varepsilon) \sim (\varepsilon - \varepsilon_F)^2 \ln |\varepsilon - \varepsilon_F|$  as  $\varepsilon \rightarrow \varepsilon_F$ . Since the leading  $\varepsilon$ -dependent contribution to  $\text{Re}\Sigma(k = k_F; \varepsilon)$  associated with this  $\text{Im}\Sigma(k = k_F; \varepsilon)$  is proportional to  $(\varepsilon - \varepsilon_F)$  (see Appendix C), we have that  $\Sigma(k_F; \varepsilon)$  is a continuously differentiable function of  $\varepsilon$  in a neighbourhood of  $\varepsilon = \varepsilon_F$ . In view of our statements in § I (see also § IV.C), the above logarithmic contribution does *not* turn the system into a non-Fermi liquid.

<sup>33</sup> This approximation neglects the finite lifetimes of the QPs. From  $\text{Im}\Sigma(k; \varepsilon_F) \equiv 0$  (note in passing that this together with  $\varepsilon_k^0 + \hbar\Sigma(k; \varepsilon_F) = \varepsilon_F$  define the Fermi surface — see footnote 40 below) and the requirement of *continuity* of  $\text{Im}\Sigma(k; \varepsilon)$  for  $\varepsilon \rightarrow \varepsilon_F$ , we have  $\text{Im}\Sigma(k; \varepsilon) \rightarrow 0$  for  $\varepsilon \rightarrow \varepsilon_F$ . Consequently, for  $\varepsilon \rightarrow \varepsilon_F$  life-time effects are *not* of relevance to the *leading-order* contribution to  $\text{Im}\Sigma(k; \varepsilon)$  as  $\varepsilon \rightarrow \varepsilon_F$ . The mentioned 'Lorentzian' has its origin in the finite lifetimes of the QPs. For a summary of various ways in which  $\text{Im}\Sigma(k; \varepsilon)$  can vanish as  $\varepsilon \rightarrow \varepsilon_F$ , see the text following Eq. (31) in § V.

<sup>34</sup> In Luttinger's (1961) work there is no symbolic distinction between, for example, what we have denoted as  $\tilde{\Sigma}(k; z)$  and  $\Sigma(k; z)$ . Further, Luttinger does not discuss whether Eq. (18) can be exactly satisfied or not (for details see § IV.C).

<sup>35</sup> The contribution of the diagram in Fig. 1 to  $\text{Im}\Sigma(k; \varepsilon)$  for one-dimensional metals, according to Luttinger (1961, p. 946, footnote 5) amounts to  $\sim (\varepsilon - \varepsilon_F)$ , which we know to be characteristic of marginal-Fermi liquids (see Appendix C) rather than one-dimensional Luttinger liquids (see Appendix D) over a physically-relevant range of values for the anomalous dimension  $\alpha \equiv 2\gamma_0$ , that is  $0 < \alpha < 1$ . At the root of this incorrect inference by Luttinger (1961) lies the fact that in one-dimensional metals,  $\varepsilon_k$ , contrary to Luttinger's *implicit* assumption, is *not* a continuously-differentiable function of  $k$  and therefore  $d^1 k \propto d\varepsilon_k$  does *not* apply, while  $d^1 k \propto d\varepsilon_k^0$  is valid. See § IV.C.

rule out existence of states different from the Fermi-liquid state.

### C. On the dispersion of the QP energies

Here we analyse some general features of the QP energies pertinent to a uniform-electron system.

Consider Eq. (14) specialised to a uniform system. Any possible *real*-valued QP energy  $\varepsilon_k$  is a solution (which for an arbitrary  $k$  may not exist; see § V) of the following equation (see also Eq. (26) below)

$$\varepsilon_k = \varepsilon_k^0 + \hbar \Sigma(k; \varepsilon_k). \quad (18)$$

The fact that  $\tilde{\Sigma}(k; z)$  has a branch point at  $z = \mu_N \equiv \varepsilon_F := \varepsilon_{k_F}$  (§ IV.A), implies that the following equation is well satisfied by the *real*-valued energy  $\varepsilon_F$  (below  $\varepsilon_F^0 := \varepsilon_{k_F}^0$ ):<sup>36</sup>

$$\varepsilon_F = \varepsilon_F^0 + \hbar \Sigma(k_F; \varepsilon_F). \quad (19)$$

Assuming  $\Sigma(k; \varepsilon)$  to be a continuously differentiable function of  $k$  and  $\varepsilon$  in some neighbourhoods of  $k = k_F$  and  $\varepsilon = \varepsilon_F$ , respectively, from Eq. (18) together with the manifest differentiability of  $\varepsilon_k^0$  in Eq. (16) with respect to  $k$  it follows that  $\varepsilon_k$  is similarly a continuously differentiable function of  $k$  in a neighbourhood of  $k = k_F$ . Consequently,<sup>37</sup>

$$\varepsilon_k = \varepsilon_F + \hbar v_F (k - k_F) + o(k - k_F), \quad (20)$$

where

$$v_F := \frac{1}{\hbar} \left. \frac{\partial \varepsilon_k}{\partial k} \right|_{k=k_F} \equiv \frac{\hbar k_F}{m_e^*} \quad (21)$$

stands for the Fermi velocity. From Eq. (18) it can be shown that

$$\left. \frac{\partial \varepsilon_k}{\partial k} \right|_{k=k_F} = Z_{k_F} \left. \frac{\partial \{\varepsilon_k^0 + \hbar \Sigma(k; \varepsilon_F)\}}{\partial k} \right|_{k=k_F}, \quad (22)$$

where

$$Z_{k_F} := \left( 1 - \hbar \left. \frac{\partial \Sigma(k_F; \varepsilon)}{\partial \varepsilon} \right|_{\varepsilon=\varepsilon_F} \right)^{-1} \quad (23)$$

is, according to a Migdal's (1957) theorem (see § V), exactly the amount of discontinuity in the momentum distribution function  $n(p)$  at the Fermi momentum  $p_F \equiv \hbar k_F$ . Eqs. (20) and (21) are readily seen to imply that for  $k \rightarrow k_F$ ,  $\varepsilon_k \sim \hbar^2 k^2 / (2m_e^*)$ . The above steps make explicit how the latter dispersion relation is rooted in the assumption of continuous differentiability of  $\Sigma(k; \varepsilon_k)$  in a neighbourhood of  $k = k_F$ . For Fermi liquids  $0 < Z_{k_F} \leq 1$  holds,<sup>38</sup> where  $Z_{k_F} = 1$  corresponds to the case of non-interacting QPs. The condition  $Z_{k_F} = 0$ , which is a signature of marginal- and Luttinger-liquid systems, corresponds, according to Eq. (23), to the case where  $\partial \Sigma(k_F; \varepsilon) / \partial \varepsilon$  is unbounded at  $\varepsilon = \varepsilon_F$ . In cases where  $Z_{k_F}$  is vanishing, Eq. (22) *formally* implies that, for the Fermi velocity  $v_F$  to be finite,  $\partial \Sigma(k; \varepsilon_F) / \partial k$  is also to be unbounded at  $k = k_F$ . Although *formally* Eq. (20) can be maintained with  $v_F = -\hbar^{-1} \partial \Sigma(k; \varepsilon_F) / \partial k|_{k=k_F} / \partial \Sigma(k_F; \varepsilon) / \partial \varepsilon|_{\varepsilon=\varepsilon_F}$ , it should be evident, however, that in the case at hand the right-hand side of Eq. (20) neglects the fact that  $\Sigma(k; \varepsilon)$  is singular at both  $k = k_F$  and  $\varepsilon = \varepsilon_F$  with the singularities showing up already in the first derivatives, thus invalidating this formal definition of the Fermi velocity. To appreciate the significance of this observation, note that the key element in obtaining Eq. (20) has been the *assumption* of continuous differentiability of  $\Sigma(k; \varepsilon)$  in some neighbourhoods of  $k = k_F$  and  $\varepsilon = \varepsilon_F$ , whereby it has been possible

<sup>36</sup> Isotropy of the system under consideration implies  $k_F$  to be independent of the electron-electron interaction. We note in passing that this is a corollary to another celebrated theorem due to Luttinger (1960); see also Luttinger and Ward (1960).

<sup>37</sup> As indicated earlier in this work,  $o$  in  $f(x) = o(g(x))$  signifies  $f(x)/g(x) \rightarrow 0$  for  $x \rightarrow 0$ .

<sup>38</sup> Recall that  $0 < Z_{k_F} \leq 1$  although necessary, is *not* sufficient for rendering a metallic system a Fermi liquid (see § I, conditions (A) and (B)).

to write <sup>39</sup>  $\Sigma(k; \varepsilon_F) \sim \Sigma(k_F; \varepsilon_F) + \beta(k - k_F) + o(k - k_F)$  and  $\Sigma(k_F; \varepsilon) \sim \Sigma(k_F; \varepsilon_F) + \gamma(\varepsilon - \varepsilon_F) + o(\varepsilon - \varepsilon_F)$ , with  $\beta$  and  $\gamma$  *bounded*, in some neighbourhoods of  $k = k_F$  and  $\varepsilon = \varepsilon_F$ , respectively. These asymptotic relations signify the facts that  $\{\Sigma(k; \varepsilon_F) - \Sigma(k_F; \varepsilon_F)\}/(k - k_F)$  and  $\{\Sigma(k_F; \varepsilon) - \Sigma(k_F; \varepsilon_F)\}/(\varepsilon - \varepsilon_F)$  are *vanishing* for  $k \rightarrow k_F$  and  $\varepsilon \rightarrow \varepsilon_F$ , respectively. With, for instance,  $\Sigma(k; \varepsilon_F) \sim \Sigma(k_F; \varepsilon_F) + \beta|k - k_F|^{\gamma_0}$ ,  $0 < \gamma_0 < 1$  (in the present case  $\alpha=2\gamma_0$  is referred to as the *anomalous dimension*; see Appendix D, in particular paragraph following Eq. (D30)), for  $k \rightarrow k_F$ , or  $\Sigma(k_F; \varepsilon) \sim \Sigma(k_F; \varepsilon_F) + \gamma(\varepsilon - \varepsilon_F) \ln|\varepsilon - \varepsilon_F|$ , for  $\varepsilon \rightarrow \varepsilon_F$ , which have *not* been accounted for by Eq. (20),  $\{\Sigma(k; \varepsilon_F) - \Sigma(k_F; \varepsilon_F)\}/(k - k_F)$  and  $\{\Sigma(k_F; \varepsilon) - \Sigma(k_F; \varepsilon_F)\}/(\varepsilon - \varepsilon_F)$  clearly diverge for  $k \rightarrow k_F$  and  $\varepsilon \rightarrow \varepsilon_F$ , respectively, and consequently both Eq. (20) and the notion of Fermi velocity of QPs in the vicinity of the Fermi surface become altogether meaningless. <sup>40</sup>

The following two observations should be clarifying: first, any positive *non-integer*  $\gamma_0$  in the above expression would signal a non-analytic singularity in the  $k$ -dependence of  $\Sigma(k; \varepsilon)$ , even when  $\gamma_0 > 1$ ; second, even if one puts aside the possibility of non-differentiability with respect to  $k$  of  $\Sigma(k; \varepsilon)$  in a neighbourhood of  $k = k_F$ , it is certain that  $\tilde{\Sigma}(k; z)$  possesses an essential singularity at  $z = \varepsilon_F \equiv \mu_N$ . For instance, Eq. (20) with  $v_F$  defined in accordance with Eqs. (21) and (22) takes *no* account of the possibility of, for example,  $\text{Im}\Sigma(k; \varepsilon) \sim'(\varepsilon - \varepsilon_F)$  whose corresponding  $\text{Re}\Sigma(k; \varepsilon) - \Sigma(k; \varepsilon_F) \sim'(\varepsilon - \varepsilon_F) \ln|\varepsilon - \varepsilon_F|$  as  $\varepsilon \rightarrow \varepsilon_F$  (see text following Eq. (31) below as well as Appendix C) which is specific to ‘marginal’ Fermi liquids (Varma, *et al.* 1989, Littlewood and Varma 1991, Kotliar, *et al.* 1991).

Having demonstrated the difficulties that arise as a consequence of the fact that  $\Sigma(k; \varepsilon)$  (and, owing to Eq. (18),  $\varepsilon_k$ ) is not *a priori* ‘smooth’ (in our above-indicated sense), it becomes evident that the implicit assumption in the proof of the Luttinger (1961) theorem, as though  $\Sigma(k; \varepsilon)$  were invariably ‘smooth’, has *no* theoretical justification. In particular, we should like to emphasise that our above exposition makes evident that no theoretical treatments that aim to expose the behaviour of  $\Sigma(k; \varepsilon)$ , for  $\varepsilon \rightarrow \varepsilon_F$ , whether  $\Sigma(k; \varepsilon)$  is evaluated by perturbative or non-perturbative techniques, should rely on the expansion in Eq. (20), since *any* consistent treatment that relies on this expansion is bound to arriving at the conclusion that  $\Sigma(k; \varepsilon)$  is Fermi-liquid like (validity of the expansion in Eq. (20) is necessary and sufficient for the satisfaction of conditions (A) and (B) in § I); a different conclusion must of necessity signal some inconsistency (or inconsistencies) in the treatment, including plain algebraic errors.

One of the main conclusions that may be drawn from our above analyses is that a *non-Fermi-liquid-type*  $\Sigma(k; \varepsilon)$  *does not necessarily signal breakdown of the many-body perturbation theory*. Consequently, we observe that *non-Fermi-liquid behaviour can in principle be manifested in any spatial dimension higher than  $d = 1$* .

#### D. A specific feature

Here we comment on one of the specific features in the treatment by Luttinger (1961) which brings out some additional aspects concerning the above-indicated implicit assumption. The diagram in Fig. 1 has the property that it involves a so-called ‘polarisation part’, that is a part which can be disconnected from the self-energy diagram through removing two interaction lines (indicated by broken lines). Momentum conservation implies that to both of these interaction lines the same momentum-transfer vector is associated. Consequently, any polarisation part in a self-energy diagram contributes at least a *square* of the electron-electron interaction to the respective integrand. As a consequence, the contribution of the diagram in Fig. 1 is divergent when the electron-electron interaction is the bare Coulomb interaction (for a comprehensive treatment see Mattuck (1992, § 10.4)) for which  $v(\mathbf{r} - \mathbf{r}') \propto 1/|\mathbf{r} - \mathbf{r}'|$  holds and thus  $\bar{v}(k) \propto 1/k^2$  in three spatial dimensions. Thus Luttinger’s analysis is not *directly* applicable to systems

<sup>39</sup> Compare with Eq. (20). See also footnotes 6 and 37.

<sup>40</sup> We note in passing that for metallic systems the locus of the  $\mathbf{k}$ -points satisfying  $\varepsilon_k^0 + \hbar\Sigma(k; \varepsilon_F) = \varepsilon_F$  (compare with Eq. (18)) is the Fermi surface, with  $\varepsilon_k^0 + \hbar\Sigma(k; \varepsilon_F) < \varepsilon_F$  ( $> \varepsilon_F$ ) defining the interior (exterior) of the Fermi sea (Galitskii and Migdal 1958, Luttinger 1960, Eqs. (6) and (94)). Clearly, the existence of a Fermi surface is *independent* of the value of  $Z_{k_F}$ , as the latter is determined by the *derivative* with respect to  $\varepsilon$  of  $\Sigma(k_F; \varepsilon)$ ; see Eq. (23). In particular  $Z_{k_F} = 0$  does *not* rule out a Fermi surface. We point out that  $\text{Im}\Sigma(k; \varepsilon_F) \equiv 0$ , for all  $k$  (see footnote 33 above), and that in an isotropic system the Fermi surface can consist of disconnected concentric surfaces; since  $\varepsilon_k^0$  is a monotonically increasing function of  $k$ , such Fermi surfaces can exist (in isotropic systems) *only* in consequence of the electron-electron interaction: for  $\varepsilon_k^0 + \hbar\Sigma(k; \varepsilon_F) = \varepsilon_F$  to have more than one solution, it is necessary that  $\hbar\Sigma(k; \varepsilon_F)$  can counter  $\varepsilon_k^0$ . In the present work we explicitly deal with a single Fermi surface of radius  $k_F$ .

of particles interacting via the Coulomb interaction.<sup>41</sup> However, as is well known, the contribution of the set of all divergent polarisation diagrams (the random-phase approximation, RPA, bubble-like diagrams) can be exactly calculated (as these diagrams give rise to a geometric series) and one observes that screening effects in the static limit remove the singularity of the bare Coulomb interaction function  $\bar{v}(k)$  at  $k = 0$  (see, e.g., Fetter and Walecka 1971, pp. 178 and 179). In analysing the leading-order term in  $\text{Im}\Sigma(k; \varepsilon)$  as  $\varepsilon \rightarrow \varepsilon_F$ , the  $\varepsilon$ -dependence of the dynamically-screened interaction function (Hubbard 1957)  $W(\varepsilon)$  necessitates knowledge not only of the energy dispersion of the QP excitations, i.e.  $\varepsilon_k$ , but also of that of the bosonic neutral excitations, i.e.  $\mathbf{e}_k$ ; here for the energies of the ‘neutral’ excitations we have  $\mathbf{e}_s := E_{N,s} - E_{N,0} \geq 0$  (Fetter and Walecka 1971). For an interacting system, the exact dispersion of  $\mathbf{e}_k$  is unknown, similar to that of  $\varepsilon_k$ ;  $\mathbf{e}_k$  can be in principle determined from the dynamical density-density correlation function  $\chi(k; \varepsilon)$ ; for  $\mathbf{e}_k$  this function plays a similar role as  $G(k; \varepsilon)$  does for  $\varepsilon_k$ . It follows that for metallic Coulomb systems, the behaviour of  $\text{Im}\Sigma(k; \varepsilon)$  *cannot* be solely determined from the perturbation series expansion for the self-energy operator; one needs in addition the perturbation series expansion for  $\chi(k; \varepsilon)$ .

From the above arguments we conclude that, even if the implicit assumption by Luttinger (1961), discussed in §§ IV.B and IV.C, were correct, Luttinger’s asymptotic expression for  $\text{Im}\Sigma(k; \varepsilon)$  as presented in the Abstract, would *not* be justifiably applicable to self-energies of metallic systems of particles interacting via the Coulomb interaction function (see § I for references to studies on systems of particles interacting through long-range repulsive interaction functions).

## V. A BRIEF DISCUSSION OF A MIGDAL’S THEOREM

In § IV we have mentioned that the quantity  $Z_{k_F}$ , which coincides with weight of the Landau quasi-particles on the Fermi surface, is also equal to the amount of discontinuity of the momentum-distribution function  $n(p)$  at  $p_F = \hbar k_F$ . This statement constitutes a theorem due to Migdal (1957) (Luttinger 1960). Here we briefly discuss this theorem in light of our above considerations. In what follows we deal with  $n(k)$  and use the commonly-employed designation ‘momentum-distribution function’, despite the fact that  $k = p/\hbar$  is not momentum. For  $n(k)$  we have

$$n(k) := \langle \Psi_{N,0} | \hat{a}_k^\dagger \hat{a}_k | \Psi_{N,0} \rangle \equiv \frac{1}{\hbar} \int_{\mathcal{C}} \frac{dz}{2\pi i} \tilde{G}(k; z), \quad (24)$$

where  $\hat{a}_k^\dagger$  and  $\hat{a}_k$  stand for creation and annihilation operators, respectively. The contour  $\mathcal{C}$  of integration is depicted in Fig. 2. To prevent confusion, we point out that the right-hand side of Eq. (24) differs by a minus sign from the corresponding expression in Migdal’s (1957) work. This is because the single-particle Green function adopted here is defined according to the modern convention and is equal to minus the Green function employed by Migdal (1957). Further, our use of the symbol  $\tilde{G}(k; z)$  (rather than  $G(k; \varepsilon)$  which is the common notation) is in accordance with our present conventions (see § A.3) and has the advantage that our following discussions will not suffer from mathematical ambiguities. One of these ambiguities can be found in the original work by Migdal (1957). Migdal stated namely that, for  $k$  infinitesimally less than  $k_F$  (denoted by  $k_F^-$ ),  $\tilde{G}(k; z)$  would have a *pole*, with an infinitesimal imaginary part (apparently due to  $\text{Im}\Sigma(k; \varepsilon)$  as indicated in the Abstract), enclosed by  $\mathcal{C}$ . Migdal (1957) further asserted that, through increasing  $k$  from  $k_F^-$  to  $k_F^+$  ( $k_F^+$  denotes a value infinitesimally larger than  $k_F$ ), the imaginary part of the mentioned *pole* would change sign, upon which this *pole* would leave the interior of  $\mathcal{C}$ . Since the regular or *incoherent* part of  $\tilde{G}(k; z)$  does not contribute to the difference  $n(k_F^-) - n(k_F^+)$ , from Eq. (24) it would follow that  $Z_{k_F}$  would be the residue of an isolated *pole* of  $\tilde{G}(k_F^-; z)$  at  $z = \varepsilon_F$ . The Migdal theorem (Migdal 1957, Luttinger 1960) thus states

$$n(k_F^-) - n(k_F^+) = Z_{k_F}. \quad (25)$$

From the Dyson equation we have

$$\tilde{G}(k; z) = \frac{\hbar}{z - \tilde{E}_k(z)}, \quad \text{where} \quad \tilde{E}_k(z) := \varepsilon_k^0 + \hbar \tilde{\Sigma}(k; z). \quad (26)$$

*Poles* of  $\tilde{G}(k; z)$  are thus seen to be the *isolated* (see § A.1) solutions of

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<sup>41</sup> Here we are referring to metallic systems. For non-metallic ground states, this problem of divergent self-energy contributions does not arise. Since non-metallic systems are by definition non-Fermi liquids, they do not concern us here.

$$\tilde{E}_k(z) = z, \quad (27)$$

which is equivalent to Eq. (14) specialised to the uniform isotropic system under consideration. According to our conventions,  $k_F^-$  has to be identified with  $k_F$  and we need only to have  $k_F^+$  for indicating a  $k$  infinitesimally larger than  $k_F$ . As we have mentioned in § IV.C, indeed  $E_{k_F}(\varepsilon_F) = \varepsilon_F$  must be satisfied. However, since  $\tilde{\Sigma}(k_F; z)$  has a *branch point* at  $z = \varepsilon_F$ , it follows that  $\tilde{G}(k_F; z)$  *cannot* have a pole at  $z = \varepsilon_F$ , since a branch point is *not* an isolated singularity (see § A.1). Moreover, as we have discussed in § III, Eq. (14), or its equivalent, Eq. (27), *cannot* have a complex-valued solution, so that there cannot be any question of a ‘pole’ of  $\tilde{G}(k; z)$  changing the sign of its imaginary part. The question, therefore, arises as to how, despite the fact that  $\tilde{G}(k_F; z)$ , or  $G(k_F; \varepsilon)$  in Migdal’s notation, has *no* pole at  $z = \varepsilon_F$ , the above-introduced Migdal (1957) theorem could hold.

To answer the above question, we note that since, for  $k = k_F$  and  $z \rightarrow \varepsilon_F$ ,  $z - \tilde{E}_k(z)$  approaches zero (by continuity and the fact that Eq. (19) applies), in principle it may be possible to write

$$\tilde{G}^{-1}(k_F; z) \equiv \frac{1}{\hbar} \{z - [\varepsilon_{k_F}^0 + \hbar \tilde{\Sigma}(k_F; z)]\} \sim \alpha(z - \varepsilon_F) + \tilde{L}(z), \quad \text{as } z \rightarrow \varepsilon_F, \quad (28)$$

with  $\tilde{L}(z)$  satisfying  $\lim_{z \rightarrow \varepsilon_F} \tilde{L}(z)/(z - \varepsilon_F) = 0$ , that is  $\tilde{L}(z) = o(z - \varepsilon_F)$  for  $z \rightarrow \varepsilon_F$ .<sup>42</sup> In Eq. (28),  $\alpha$  stands for a constant to be specified below. Equation (28) implies  $\tilde{G}^{-1}(k_F; z)$ , and thus  $\tilde{\Sigma}(k_F; z)$ , to be at least once *continuously differentiable* with respect to  $z$  at  $z = \varepsilon_F$  (see footnote 9);  $m$ th continuous differentiability of  $\tilde{G}^{-1}(k_F; z)$ , and thus of  $\tilde{\Sigma}(k_F; z)$ , with respect to  $z$  at  $z = \varepsilon_F$  amounts to the condition  $\tilde{G}^{-1}(k_F; z) \sim \sum_{j=0}^m \alpha_j (z - \varepsilon_F)^j + \tilde{L}(z)$  with  $\{\alpha_j\}$  finite constants and  $\lim_{z \rightarrow \varepsilon_F} \tilde{L}(z)/(z - \varepsilon_F)^m = 0$ .<sup>43</sup> Consequently,  $\partial \Sigma(k_F; \varepsilon)/\partial \varepsilon|_{\varepsilon=\varepsilon_F}$  is well-defined and one has  $\alpha \equiv \hbar^{-1}(1 - \hbar \partial \Sigma(k_F; \varepsilon)/\partial \varepsilon|_{\varepsilon=\varepsilon_F}) \equiv 1/(\hbar Z_{k_F})$  (see Eq. (23) above). Using the representation  $z - \varepsilon_F = \varrho \exp(i\varphi)$ , with  $\varrho$  a non-vanishing constant, to be let to approach zero, from Eq. (24) it can straightforwardly be shown that Eq. (25) remains intact, even though  $z = \varepsilon_F$  is not a *pole* (i.e. an isolated singularity) of  $\tilde{G}(k; z)$ .

Through the Kramers-Kronig relation (see Appendices B and C) for  $\text{Re}\Sigma(k; \varepsilon)$  in terms of  $\text{Im}\Sigma(k; \varepsilon)$  and the asymptotic relation  $\text{Im}\Sigma(k; \varepsilon) \sim \mp \alpha_k (\varepsilon - \varepsilon_F)^2$ , for  $\varepsilon \gtrless \varepsilon_F$ , it can readily be deduced that (see Appendix C)

$$\text{Re}\Sigma(k; \varepsilon) \sim \Sigma(k; \varepsilon_F) + \beta_k (\varepsilon - \varepsilon_F), \quad \text{with } \beta_k \leq 0 \text{ for } k \rightarrow k_F. \quad (29)$$

Concerning the sign assigned to  $\beta_k$  for  $k$  close to  $k_F$ , as is seen from Eq. (31) below, a positive  $\beta_{k_F}$  would imply  $Z_{k_F} > 1$  which is impossible on account of a combination of the following three reasons.

- (i) By definition (see Eq. (24) above)  $0 \leq n(k) \leq 1$ ; this can also be traced back to the Pauli exclusion principle.
- (ii) According to Eq. (25),  $Z_{k_F} = n(k_F^-) - n(k_F^+)$  so that in view of (i),  $|Z_{k_F}| \leq 1$ .
- (iii)  $Z_{k_F} < 0$  is excluded owing to  $\text{Im}\Sigma(k; \varepsilon) \lesseqgtr 0$ , for  $\varepsilon \gtrless \mu$ .

In Appendix C (see text following Eq. (C9)) we demonstrate that  $\beta_k \leq 0$ , for  $k$  in a neighbourhood of  $k_F$ , directly follows from a general sum rule involving  $\text{Im}\Sigma(k; \varepsilon)$ . We therefore have<sup>44</sup>

<sup>42</sup> Because of  $\tilde{G}(z^*) = \tilde{G}^\dagger(z)$ , it follows that  $\tilde{L}(z^*) = \tilde{L}^*(z)$ . Further, unless  $\varepsilon \in (\mu_N, \mu_{N+1})$ , in general  $\lim_{\eta \downarrow 0} \{\tilde{L}(\varepsilon + i\eta) - \tilde{L}(\varepsilon - i\eta)\} \neq 0$ . This is a consequence of the fact that  $z = \mu_N \equiv \varepsilon_F$  and  $z = \mu_{N+1}$  are branch points of  $\tilde{G}(k; z)$ , as discussed in § III.

<sup>43</sup> We note that, when a function, say  $\tilde{f}(z)$ , is analytic inside and on the boundary of a *simply-connected* region  $\mathcal{R}$  of the complex  $z$ -plane, it is infinitely-many times differentiable in  $\mathcal{R}$ , so that once-differentiability of  $\tilde{f}(z)$  at a point, say  $z_0$ , interior to  $\mathcal{R}$  implies infinitely-many times differentiability of  $\tilde{f}(z)$  at  $z = z_0$ . In the case under consideration, since  $z = \varepsilon_F$  is a branch point, it *cannot* be interior to any *simply-connected* region of analyticity of  $\tilde{G}(k_F; z)$  (see § A.1). Consequently,  $m$ th differentiability of this function at  $z = \varepsilon_F$  does *not* imply its  $(m+1)$ th differentiability at this point. Consider for instance  $\tilde{f}(z) := (z - z_0)^p$  with  $p > 0$  a real non-integer. This function has two branch points,  $z = z_0$  and  $1/z = 0$ . With  $m$  an integer satisfying  $m < p < m+1$ ,  $\tilde{f}(z)$  is seen to be  $m$  times, but *not*  $m+1$  times, differentiable at  $z = z_0$ .

<sup>44</sup> Consider the analytic function  $\tilde{f}(z) \equiv u(x, y) + iv(x, y)$ , where  $x + iy \equiv z$  and  $u(x, y)$  and  $v(x, y)$  are real. Upon setting  $y$  equal to zero, we obtain  $\tilde{f}(x) = u(x, 0) + iv(x, 0)$ . Replacing  $x$  in the argument of  $\tilde{f}$  by  $z$ , we obtain  $\tilde{f}(z) = u(z, 0) + iv(z, 0)$  (in a similar manner one obtains  $\tilde{f}(z) = u(0, -iz) + iv(0, -iz)$ ). This expression prescribes how to obtain  $\tilde{\Sigma}(k; z)$  from the knowledge of  $\Sigma(k; \varepsilon)$ . In the present case, as  $z \rightarrow \varepsilon_F$ , we have:  $\tilde{\Sigma}(k; z) \sim \Sigma(k; \varepsilon_F) + \beta_k (z - \varepsilon_F) - i\alpha_k (z - \varepsilon_F)^2$ , for  $0 \leq \arg(z) < \pi$ , and  $\tilde{\Sigma}(k; z) \sim \Sigma(k; \varepsilon_F) + \beta_k (z - \varepsilon_F) + i\alpha_k (z - \varepsilon_F)^2$ , for  $-\pi \leq \arg(z) < 0$ . The self-energy as specified in this way is seen to preserve the reflection property  $\tilde{\Sigma}(k; z^*) = \tilde{\Sigma}^*(k; z)$  for  $\text{Im}(z) \neq 0$  (see Eq. (12)).



$$\Sigma(k; \varepsilon) \sim \Sigma(k; \varepsilon_F) + \beta_k(\varepsilon - \varepsilon_F) \mp i\alpha_k(\varepsilon - \varepsilon_F)^2, \quad \varepsilon \gtrless \varepsilon_F. \quad (30)$$

A simple calculation reveals that for  $Z_{k_F}$  in Eq. (25) (or Eq. (28))

$$Z_{k_F} \equiv (1 - \hbar\beta_{k_F})^{-1} \quad (31)$$

holds. A similar analysis based on the Kramers-Kronig relation for the self-energy (see Appendix C) yields that when<sup>45</sup>  $\text{Im}\Sigma(k; \varepsilon) \sim |\varepsilon - \varepsilon_F|^\sigma$  for  $\varepsilon \rightarrow \varepsilon_F$ , the following hold.

- (i)  $\text{Re}\Sigma(k; \varepsilon) - \Sigma(k; \varepsilon_F) \sim'(\varepsilon - \varepsilon_F)$  when  $1 < \sigma \leq 2$ .
- (ii)  $\text{Re}\Sigma(k; \varepsilon) - \Sigma(k; \varepsilon_F) \sim'(\varepsilon - \varepsilon_F) \ln|\varepsilon - \varepsilon_F|$  when  $\sigma = 1$  (“marginal Fermi liquid” (Varma, *et al.* 1989, Littlewood and Varma 1991, Kotliar, *et al.* 1991) (see Appendix C).
- (iii)  $\text{Re}\Sigma(k; \varepsilon) - \Sigma(k; \varepsilon_F) \sim'|\varepsilon - \varepsilon_F|^\sigma$  when  $0 < \sigma < 1$  (see § IV.C, text following Eq. (23), and identify  $\sigma$  with  $\gamma_0$ ; see also Appendix D and identify  $\sigma$  with  $1 - 2\gamma_0$  in Eqs. (D21) and (D22) and with  $1 - \gamma_0$  in Eqs. (D28) and (D29)).

It can straightforwardly be shown (using the same strategy as in the case of  $\sigma = 2$  employed above) that, for  $0 < \sigma \leq 1$ ,  $Z_{k_F}$  on the right-hand side of Eq. (25) is vanishing, whereas for  $1 < \sigma \leq 2$  it takes a finite value. Vanishing of  $Z_{k_F}$  corresponds to disappearance of the Landau QPs on the Fermi surface. Although for  $1 < \sigma \leq 2$ ,  $Z_{k_F}$  is non-vanishing, it is evident that the smaller the  $\sigma$ , the shorter are the life-times of the QPs close to the Fermi surface, if QPs can at all be meaningfully defined (recall condition (B) introduced in § I). We should emphasize that as  $\tilde{G}(k_F; z)$  does *not* possess an isolated pole at  $z = \varepsilon_F \equiv \mu_N$ , but a branch point, it follows that even in the case of the Landau Fermi liquids, the QPs on the Fermi surface are not truly infinitely long-lived but only so in an asymptotic sense.

We conclude that what Migdal (1957) describes as ‘leaving of a pole from the interior of contour  $\mathcal{C}$  upon changing  $k$  from  $k_F^-$  to  $k_F^+$ ’ is to be understood as follows: for  $k = k_F^+$ , Eq. (18) (or Eq. (27)) *cannot* be satisfied when  $z = \varepsilon_F \equiv \mu_N$ , that is  $E_{k_F^+}(\mu_N) \neq \mu_N$ ; rather we have  $E_{k_F^+}(\mu_{N+1}) = \mu_{N+1}$  and, as can be viewed from Fig. 2,  $z = \mu_{N+1}$  indeed does not lie in the interior of  $\mathcal{C}$ .

In closing this Section, we indicate the following observation. In §§ I and IV.C we argued that Fermi liquids are distinguished by *two* specific aspects, namely that of condition (A), that is continuous differentiability of  $\Sigma(k_F; \varepsilon)$  with respect to  $\varepsilon$  in a neighbourhood of  $\varepsilon = \varepsilon_F$ , and of condition (B), that is continuous differentiability of  $\Sigma(k; \varepsilon_F)$  with respect to  $k$  in a neighbourhood of  $k = k_F$ . It is owing to these that the Landau quasi-particle dispersion as presented in Eq. (20) can be rigorously obtained from the quasi-particle equation, Eq. (18), which is applicable to *all* isotropic metallic systems of spin-less fermions; conditions (A) and (B) are prerequisite to assigning a *unique* mass as well as velocity to an elementary excitation at and in a close vicinity of the Fermi surface (see Eq. (21) above). However, as we have observed in the present Section, in obtaining Eq. (25), only Eq. (28) combined with the condition  $\lim_{z \rightarrow \varepsilon_F} \tilde{L}(z) = 0$  (which together embody aspect (A)) played a role. Therefore, a non-vanishing  $Z_{k_F}$  which is universally considered as *the* hallmark for Fermi liquids, is only a *necessary* condition for a metallic system to qualify as a Fermi liquid, but by no means a *sufficient* condition (see § I). In other words, metallic systems with  $Z_{k_F} \neq 0$  can in principle be non-Fermi liquids in the above sense.

## VI. (NON-)FERMI LIQUIDS AND NON-DIFFERENTIABILITY OF THE SELF-ENERGY WITH RESPECT TO MOMENTUM CLOSE TO AND ON THE FERMI ‘SURFACE’

From Eq. (24), making use of Eq. (26), one obtains

$$\frac{\partial n(k)}{\partial k} = \int_{\mathcal{C}} \frac{dz}{2\pi i} \frac{1}{(z - \tilde{E}_k(z))^2} \frac{\partial \tilde{E}_k(z)}{\partial k}. \quad (32)$$

In view of the singular behaviour of  $n(k)$  at  $k = k_F$ , in employing Eq. (32) we approach  $k_F$  from either left or right, i.e. by  $\partial n(k)/\partial k|_{k=k_F^\mp}$  we imply left and right derivatives, respectively.

For our further discussions it is convenient to introduce some notational conventions. We denote the circular contour circumscribing  $\varepsilon_F$  in Fig. 2 by  $\mathcal{C}_s$  (subscript  $s$  refers to ‘singular’, in view of the singularity of  $\tilde{G}(k_F; z)$  at  $z = \varepsilon_F$ ) and its complement,  $\mathcal{C} \setminus \mathcal{C}_s$ , by  $\mathcal{C}_r$  (subscript  $r$  refers to ‘regular’). Consequently, we write  $n(k) = n_s(k) + n_r(k)$  with  $n_s(k)$

<sup>45</sup> Recall that  $\sim'$  indicates that the corresponding asymptotic relation is correct up to a multiplicative constant.

and  $n_r(k)$  corresponding to the  $z$ -integrals along  $\mathcal{C}_s$  and  $\mathcal{C}_r$ , respectively (here we assume the radius  $\varrho$  characterising  $\mathcal{C}_s$  to be infinitesimally small).

In arriving at the result in Eq. (25), we have made use of  $n_r(k_F^+) = n_r(k_F^-)$  so that Eq. (25) in fact amounts to  $n_s(k_F^-) - n_s(k_F^+) = Z_{k_F}$ . A similar property, namely  $\partial n_r(k)/\partial k|_{k=k_F^-} = \partial n_r(k)/\partial k|_{k=k_F^+}$ , may not be applicable in general, in particular since  $\partial n_r(k)/\partial k|_{k=k_F^\mp}$  may be unbounded. Below we demonstrate that:<sup>46</sup> i) for Fermi liquids  $\partial n(k)/\partial k|_{k=k_F^-}$  is *either* finite, in which case it follows that  $\tilde{\Sigma}(k; z)/\partial k|_{k=k_F^-}$  is finite for *all*  $z$ , or infinite, in which case  $\partial \beta_k/\partial k|_{k=k_F^-}$  is also infinite and consequently  $\partial\{\tilde{\Sigma}(k; z) - \Sigma(k; \varepsilon_F)\}/\partial k|_{k=k_F^-}$  is infinite for *all*  $z$  (note that, for Fermi liquids,  $\Sigma(k; \varepsilon_F)$  is *by definition* a continuously differentiable function of  $k$  in a neighbourhood of  $k_F$ ); ii) for non-Fermi liquids characterised by  $Z_{k_F} = 0$ ,  $\partial n_s(k)/\partial k|_{k=k_F^-} = 0$ . Further, although for these systems  $n(k_F^-) - n(k_F^+) = 0$  holds,  $\partial n(k)/\partial k$  may or may not diverge as  $k \uparrow k_F$  ( $k \downarrow k_F$ ); in the one-dimensional Luttinger model (Luttinger 1963, Mattis and Lieb 1965), which is a prototype of one-dimensional systems of interacting spin-less fermions (Haldane 1980, 1981), with ‘weak’ interaction — as characterised by a small anomalous dimension,<sup>47</sup> namely  $0 < \alpha < 1$  —,  $\partial n(k)/\partial k$  does diverge for  $k \rightarrow k_F^\mp$ , and consequently  $\partial \tilde{\Sigma}(k; z)/\partial k|_{k=k_F^-}$  is infinite for *all*  $z$ ; here, contrary to the case of Fermi liquids,  $\partial \Sigma(k; \varepsilon_F)/\partial k|_{k=k_F^-}$  is *not* necessarily finite. This aspect can be explicitly verified from the available explicit expression for the spectral function corresponding to the retarded single-particle Green function (see Appendix D).

Now we proceed with demonstrating the above statements. In cases where Eq. (28) and  $\lim_{z \rightarrow \varepsilon_F} \tilde{L}(z) = 0$  are satisfied, making use of the Cauchy residue theorem, one obtains

$$\left. \frac{\partial n_s(k)}{\partial k} \right|_{k=k_F^-} = \hbar Z_{k_F}^2 \left. \frac{\partial^2 \Sigma(k; \varepsilon)}{\partial \varepsilon \partial k} \right|_{\substack{\varepsilon=\varepsilon_F \\ k=k_F^-}}, \quad \left. \frac{\partial n_s(k)}{\partial k} \right|_{k=k_F^+} = 0. \quad (33)$$

In view of the derivative with respect to  $\varepsilon$ ,  $\Sigma(k; \varepsilon)$  in Eq. (33) may be replaced by  $\Sigma(k; \varepsilon) - \Sigma(k; \varepsilon_F)$ . From this it follows that a possible divergence of  $\partial n_s(k)/\partial k$  for  $k \uparrow k_F$  *cannot* be due to  $\Sigma(k; \varepsilon_F)$  and therefore a divergent  $\partial n_s(k)/\partial k$  for  $k \uparrow k_F$  is *not* in conflict with the fundamental assumption of the Fermi-liquid theory, namely that of continuous differentiability of  $\Sigma(k; \varepsilon_F)$  in a neighbourhood of  $k = k_F$ . We note in passing that, through employing Eqs. (30) and (31), we can also write  $\partial n_s(k)/\partial k|_{k=k_F^-} = \partial Z_k/\partial k|_{k=k_F}$ , where  $Z_k := (1 - \hbar \beta_k)^{-1}$ .<sup>48</sup> In Appendix C we demonstrate that divergence of  $\partial \beta_k/\partial k$  (which in view of the latter result implies that of  $\partial n_s(k)/\partial k$ ) at any  $k = k_0$ , e.g.  $k = k_F$ , signals divergence of  $\partial \tilde{\Sigma}(k; z)/\partial k|_{k=k_0}$  for *all*  $z$ ,  $\text{Im}(z) \neq 0$ ; in Appendix D we explicitly show this to be the case for the (one-dimensional) Luttinger model for spin-less fermions. With reference to Eq. (32), as well as Eq. (26) where  $\tilde{E}_k(z)$  is defined in terms of  $\tilde{\Sigma}(k; z)$ , divergence of  $\partial n_s(k)/\partial k$  at  $k = k_F^-$  is thus seen to imply divergence of  $\partial n(k)/\partial k$  at  $k = k_F^-$ . This completes demonstration of our point (i) above. It is relevant to mention that for a three-dimensional isotropic system of fermions interacting through a hard-core potential, qualifying as a Fermi liquid,  $\partial n(k)/\partial k$  has been shown to be logarithmically divergent for  $k \rightarrow k_F^\mp$  (Belyakov 1961, Sartor and Mahaux 1980).<sup>49</sup> On the other hand, for a similar system of electrons, interacting through the long-range Coulomb interaction, no divergence is observed in  $\partial n(k)/\partial k$ ,  $k \rightarrow k_F^\mp$ , within the framework of the random-phase approximation (RPA) (Daniel and Vosko 1960); calculations of  $n(k)$  beyond the RPA are also available (Geldart, Houghton and Vosko 1964), however the results do not provide information with regard to regularity or otherwise of  $\partial n(k)/\partial k$  at  $k = k_F^\mp$ .<sup>50</sup>

For non-Fermi liquids, such as the marginal Fermi liquid considered in Appendix C, the result  $\partial n_s(k)/\partial k|_{k=k_F^-} = 0$  follows exactly for the same reason that  $n_s(k_F^-) = 0$  for these systems: through parametrising  $\mathcal{C}_s$  in terms of the circular-polar angle  $\varphi$ , one observes that despite singularity of the integrand at  $\varepsilon_F$ , the  $\varphi$ -integration over  $[-\pi, \pi)$  yields an identically-vanishing contribution. As our considerations in § V have made explicit, at the root of this

<sup>46</sup> Similar considerations can be given for the case  $k = k_F^+$ .

<sup>47</sup> The non-interacting case corresponds to  $\alpha \equiv 2\gamma_0 = 0$ . See § IV.C and Appendix D.

<sup>48</sup> We should emphasise that here  $Z_k$  is merely a generalisation of  $Z_{k_F}$  and does not necessarily have the same physical significance as  $Z_{k_F}$  does. To appreciate the relevance of this remark, one should realise that for  $k \neq k_F$ , Eq. (18) may *not* have a real-valued solution.

<sup>49</sup> In our judgement, nothing precludes the possibility that this divergence is an artifact of the second-order perturbation theory employed in the treatment by Belyakov (1961).

<sup>50</sup> For a calculation of  $n(k)$  at large  $k$  as well as some general discussions concerning  $n(k)$  see also Yasuhara and Kawazoe (1976).

property lies the non-differentiability with respect to  $\varepsilon$  of  $\Sigma(k_F; \varepsilon)$  at  $\varepsilon_F$ . We emphasise that this is *sufficient* for disqualifying a system to be a Fermi liquid; our discussions in § IV.C have made evident that this non-differentiability condition stands in the way of obtaining the quasi-particle energy dispersion in Eq. (20) from Eq. (18).

For a *general* non-Fermi-liquid metallic system, in *arbitrary*  $d$ -dimensional spatial space (if indeed such a system exists in  $d > 1$ ), our knowledge with regard to behaviour of  $\partial n(k)/\partial k$  in some neighbourhoods of  $k = k_F^\mp$  cannot surpass that of  $\partial n(k)/\partial k$  corresponding to Fermi-liquid systems.<sup>51</sup> Since, by definition, to a non-Fermi-liquid metallic system *no* quasi-particle energy dispersion similar to that in Eq. (20) can correspond (see above), and this can be effected through a non-continuous differentiable  $\Sigma(k_F; \varepsilon)$  in a neighbourhood of  $\varepsilon = \varepsilon_F$ , the non-continuous-differentiability with respect to  $k$  of  $\Sigma(k; \varepsilon_F)$  pertaining to these systems cannot in general be ruled in but also not ruled out. In  $d = 1$  dimension, the exactly-solvable Luttinger (1963) model provides us with the opportunity to arrive at a rigorous statement, however. In this model, for  $k$  close to  $k_F$  one has (see, e.g., Voit 1993b)  $n(k) \sim 1/2 - C_1 \text{sgn}(k - k_F)|k - k_F|^\alpha + C_2(k - k_F)$  where  $C_1$  and  $C_2$  are constants and  $\alpha \equiv 2\gamma_0$ , the ‘anomalous dimension’, is not universal but depends on the nature and strength of the particle-particle interaction in the system under investigation; for the one-dimensional Hubbard model corresponding to a finite repulsive on-site interaction,  $U$ , and away from the half-filling of the Hubbard band,  $0 < \alpha < 1/8$ , with the upper limit achieved for infinitely large  $U$ . One observes that for  $0 < \alpha < 1$ , which in view of the latter observation should amount to a wide range of values for  $\alpha$ ,  $n(k)$ , although continuous, is *not* continuously differentiable in a neighbourhood of  $k = k_F$ ;  $\partial n(k)/\partial k$  diverges as  $k \rightarrow k_F^\mp$ .

## VII. SUMMARY AND CONCLUDING REMARKS

In this work we have presented a critical analysis of a celebrated theorem due to Luttinger (1961) concerning energies and life-times of low-energy QP excitations in interacting systems. This theorem has played the crucial role of classifying *all* metallic systems in spatial dimensions larger than unity as Landau Fermi liquids, with the explicit assumption with regard to applicability of the many-body perturbation theory to all orders to these systems.<sup>52</sup> In this work we have demonstrated that *Luttinger’s (1961) proof involves an implicit assumption* with regard to the dispersion of the QP energies, which assumption we have explicitly shown to be specific to Fermi-liquid systems. We have therefore shown that Luttinger’s proof amounts to a *demonstration of consistency* of the mentioned implicit assumption with the property  $\text{Im}\Sigma(k; \varepsilon) \sim (\varepsilon - \varepsilon_F)^2$ , for  $\varepsilon \rightarrow \varepsilon_F$ , to *all* orders of the many-body perturbation theory. It follows that, contrary to the commonly-held view, the many-body perturbation theory does *not* necessarily break down when applied to systems whose self-energies are of non-Fermi-liquid type. In the absence of any theorem to replace the Luttinger theorem, one could reasonably conjecture that it is most likely that, in particular in two spatial dimensions, non-Fermi-liquid-like metallic systems if not abundant, should not be rare and that these may be correctly addressed within the framework of the many-body perturbation theory. In this connection it is important to point out the following two observations.

First, *any* static and local approximation to the self-energy operator implies a Fermi-liquid behaviour for the QPs, no matter how ingenious such approximation may be. The entire body of the *energy-band* methods based on the *conventional* density-functional theory, involving a local and energy-independent effective potential (for a comprehensive review see Dreizler and Gross 1990), pertains to this category of approximation frameworks. The (self-consistent) Hartree-Fock scheme which involves the non-local static exchange self-energy, *may* describe non-Fermi liquid metals, however only those whose  $Z_{k_F} = 1$ .<sup>53</sup> Second, in evaluating the self-energy operator for a Coulomb system, within the framework of the many-body perturbation theory, it has to be realised that in metallic systems the

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<sup>51</sup> Here we employ the notion of ‘neighbourhood’ in a loose sense, since, as we have mentioned earlier, here by  $\partial n(k)/\partial k$ ,  $k \rightarrow k_F$ , we express *left* or *right* derivative of  $n(k)$ , thus excluding  $k = k_F$  from the interior of the interval over which  $n(k)$  is being differentiated.

<sup>52</sup> This on account of the fact that the proof of this theorem has been based on an infinite-order perturbation expansion of the self-energy operator.

<sup>53</sup> For uniform systems of electrons interacting through the Coulomb potential, the derivative with respect to  $k$  of the Hartree-Fock self-energy is logarithmically divergent at  $k = k_F$  (Ashcroft and Mermin 1981, p. 334) — see Appendix B. Since  $\Sigma^{HF}$  is independent of  $\varepsilon$  (Appendix B), it is therefore analytic over the entire  $z$ -plane and thus the associated  $Z_{k_F} = 1$  (see § V). Therefore a metal within the Hartree-Fock scheme may be non-Fermi liquid solely on account of violation of condition (B) introduced in § I.

long range of the *bare* Coulomb interaction gives rise to divergent self-energy contributions (from the second order in the interaction onwards). Thus, for such systems, perturbation expansion has to be in terms of the dynamically-screened electron-electron interaction function. In calculating this function, at least *all* polarisation diagrams of the random-phase approximation have to be taken into account, for all these diagrams have unbounded contributions and only in combination give rise to a finite result. Our considerations with regard to the problem of unbounded self-energy diagrams have led us to draw the additional conclusion that, irrespective of our above-indicated finding, Luttinger's (1961) proof *cannot* have bearing on metallic systems of electrons interacting through the long-range Coulomb interaction function. This follows from the fact that Luttinger's proof has been based on a perturbation expansion of the self-energy operator in terms of the *bare* electron-electron interaction. In this connection we point out that analytic properties of a series of functions of complex variable do not coincide with those of the constituent terms, unless the series be *uniformly* convergent (Whittaker and Watson 1927, pp. 91 and 92, Titchmarsh 1939, pp. 95-98). Further, the sum of a series which is not *absolutely convergent* depends on the order of summation of terms in the series (Whittaker and Watson 1927, pp. 18 and 25). It is evident that, in particular, perturbation series which involve singular terms *cannot* be absolutely or uniformly convergent.

We have further elaborated upon the analytic properties of the single-particle Green function and the self-energy operator as functions of a complex-valued energy parameter,  $z$ . In particular we have pointed out that the commonly-used equation for energies of the QPs in an interacting system can yield, if any, only real-valued solutions. To obtain complex-valued solutions (concerning systems in the thermodynamic limit), the self-energy operator has to be analytically continued into a *non-physical* RS of the complex energy plane.

We have paid careful attention to the precise nature of the singular points of the single-particle Green function and the self-energy operator. For instance we have pointed out that the Fermi energy is not an *isolated singularity* of the Green function and therefore *cannot* be a pole. In the light of this, we have considered a celebrated theorem due to Migdal (1957) and discussed how its proof is independent of the commonly-made assumption, that on the Fermi surface the Fermi energy were a *pole* of the single-particle Green function. We have explicitly shown that for the momentum distribution function  $n(k)$  pertaining to a metallic system to be discontinuous at  $k = k_F$  by a finite amount — the magnitude of this discontinuity being, according to the Migdal theorem, equal to  $Z_{k_F}$  —, it is only necessary that the self-energy  $\Sigma(k_F; \varepsilon)$  be a continuously differentiable function of  $\varepsilon$  in a neighbourhood of  $\varepsilon = \varepsilon_F$ . Consequently metallic systems with  $Z_{k_F} \neq 0$  are *not* necessarily Fermi liquids. Fermi liquids, we have shown, have in addition the property that their corresponding  $\Sigma(k; \varepsilon_F)$  is a continuously differentiable function of  $k$  in a neighbourhood of  $k = k_F$ . In spite of this,  $\Sigma(k; \varepsilon)$  pertaining to Fermi liquids may not be continuously differentiable function of  $k$  in a neighbourhood of  $k = k_F$  when  $\varepsilon \neq \varepsilon_F$ . These observations, which we have explicitly examined on the self-energies of Fermi-, marginal Fermi- and Luttinger-liquids, clearly demonstrate how uncritical Taylor expansions of various functions of  $k$  and  $\varepsilon$  associated with the self-energy can lead to incorrect description of the physical properties of the systems under consideration at low energies.

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## APPENDIX A: MATHEMATICAL PRELIMINARIES

Since in the present work we repeatedly encounter a number of specific mathematical notions, we devote this Appendix to a brief exposition of these.

## 1. On the types of singularity

A point at which a function  $g(z)$  of complex variable  $z$  is not *analytic*<sup>54</sup> is called a *singular* point (for details of what follows see, for example, Whittaker and Watson 1927, pp. 102 and 104, Titchmarsh 1939, pp. 89-95). Such a point is either *isolated* or *non-isolated*;  $z_0$  is isolated if there exists a  $\delta > 0$  such that inside the region  $|z - z_0| < \delta$  there exists no other singular point of  $g(z)$  than  $z_0$ . Otherwise  $z_0$  is not isolated.

A singularity may be *removable*; such singularity corresponds to a point  $z_0$  at which  $g(z)$  is not defined, but  $\lim_{z \rightarrow z_0} g(z)$  exists. Thus  $z = 0$  is a removable singularity of  $g(z) := \sin(z)/z$ .

*Limiting* (or *accumulation*) point of an infinite sequence of poles, is *not* classified as a pole and thus is considered as an *essential* singularity, and here, a non-isolated essential singularity. For instance, the sequence of poles of  $g(z) := \sum_{n=0}^{\infty} 1/(n![1 + a^{2n}z^2])$ , with  $a > 1$ , have  $z = 0$  as their limiting point which is not isolated. This function has *no* Taylor or Laurent series expansion (see § A.2) over any domain of the  $z$ -plane which has  $z = 0$  as its *interior*.

Let  $g(z)$  be a single-valued function throughout a domain  $\mathcal{D}$  at whose interior point  $z_0$ ,  $g(z)$  is singular. Suppose that the *principal part* of the Laurent series expansion (see § A.2) of  $g(z)$  around  $z = z_0$  terminates with the term  $a_{-n}/(z - z_0)^n$ , with  $a_{-n}$  a non-vanishing constant. In such case,  $z_0$  is called a *pole* of order  $n$ . Poles are thus by definition *isolated* singularities.

If the principal part of the Laurent expansion of  $g(z)$  around  $z = z_0$  does not terminate (i.e., if there exists *no*  $n_0$  such that  $a_{-n} = 0$  for all  $n > n_0$ ),  $z_0$  is an *isolated* essential singularity of  $g(z)$ . The function  $g(z) := \exp(z)$  has one such singularity at the point of infinity, i.e. at  $z = 1/\zeta$  where  $\zeta = 0$ .

*Branch points* belong to the class of singular points and concern multi-valued functions. Let  $g(z)$  be one such function. By traversing a closed contour which circumscribes only one branch point of  $g(z)$ , one obtains, upon arriving at the starting point  $z_1$ , a different value for  $g(z_1)$ , indicating change of the initial branch of  $g(z)$  into a different branch; for a branch point of order  $p$ , the original branch is recovered after completion of  $p$  revolutions along the mentioned contour. Thus  $(z - z_0)^{1/3}$  has a third-order branch point at  $z = z_0$ . Functions can also possess branch points of infinite order; for  $g(z) := \ln(z)$ ,  $z = 0$  and  $1/z = 0$  are such points.

## 2. Taylor, Laurent and asymptotic series

In the Abstract of this work we have referred to the expression  $\text{Im}\Sigma(k; \varepsilon) \sim \mp \alpha_k (\varepsilon - \varepsilon_F)^2$ ,  $\varepsilon \gtrless \varepsilon_F$ , for  $\varepsilon \rightarrow \varepsilon_F$ , as an *asymptotic* relation. Here we specify what asymptotic relations and asymptotic series are and in what essential respects these series differ from the Taylor and the Laurent series (Whittaker and Watson 1927, pp. 93, 94 and 100). We also indicate the interrelation between branch points (see § A.1) and divergent asymptotic series.

When  $g(z)$  is analytic at  $z = z_0$ , then, by definition, there exists an *open* domain  $\mathcal{D}$  of which  $z_0$  is an interior and over which  $g(z)$  is analytic. Within a circle around  $z_0$  embedded within  $\mathcal{D}$ ,  $g(z)$  can be represented in terms of a Taylor series:  $g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ , with *unique* coefficients  $a_n$ . Thus  $\{(z - z_0)^n | n = 0, 1, \dots\}$  can be considered as a *complete* basis in a (finite) neighbourhood of  $z_0$  for *any* function that is analytic at  $z = z_0$  (note that this set does not contain such term as, for instance,  $(z - z_0)^\sigma$  with  $\sigma$  non-integer). This basis can be extended to  $\{(z - z_0)^n | n = -m, -m + 1, \dots, 0, 1, \dots\}$  in order to form a *complete* basis in a (finite) neighbourhood of  $z_0$  for representing *any* function which has a pole of order  $m$  or lower at  $z_0$  and is analytic in a neighbourhood of  $z = z_0$ ; the resulting representation is the well-known Laurent series expansion (§ A.1). It *cannot* be extended, through increasing  $m$  in  $\{(z - z_0)^n | n = -m, -m + 1, \dots, 0, 1, \dots\}$ , for representing functions that possess a branch point at  $z = z_0$  (see § A.3). It is important to point out that the Taylor and the Laurent series expansions are very special in that they provide *uniform* representations of an analytic function; when these series are convergent for any  $\varrho$  and  $\varphi$  in  $z - z_0 \equiv \varrho \exp(i\varphi)$ , they are *uniformly* convergent for *all* values of  $\varphi$ ,  $\varphi \in [0, 2\pi)$ . Branch points are peculiar in that they do not allow for *any* uniform representation (note the ‘ $\mp$ ’ in the above expression for  $\text{Im}\Sigma(k; \varepsilon)$ ). We shall return to this point further on in this Section.

Asymptotic series (in the sense of Poincaré) (Whittaker and Watson 1927, Ch. VIII, Copson 1965, Dingle 1973, Lauwerier 1977) are more general than the Taylor and the Laurent series in that they are in terms of “basis functions” that are not necessarily of the form  $(z - z_0)^n$ , with  $n$  possibly negative, zero and positive integer. To an asymptotic series corresponds a so-called *asymptotic sequence*: The set  $\{\phi_n(z) | n = 0, 1, \dots\}$  is an asymptotic sequence with

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<sup>54</sup> *Analytic*, *regular* and *holomorphic* are alternative but equivalent designations.

respect to  $z_0$  provided it possesses the property  $\phi_{n+1}(z)/\phi_n(z) \rightarrow 0$ , denoted by  $\phi_{n+1}(z) = o(\phi_n(z))$ , for  $z \rightarrow z_0$ . Thus  $\{(z - z_0)^n | n = -m, -m+1, \dots, 0, 1, \dots\}$  is an asymptotic sequence with respect to  $z_0$ . We point out that some authors, for example Whittaker and Watson (1927, Ch. VIII), reserve the designation ‘asymptotic series’ for those series that are both asymptotic (in the above sense) *and* divergent. We do *not* follow this restrictive convention.

As we have mentioned above, an asymptotic series expansion of a function may not be convergent. It may not also be *uniform*; functions with non-uniform asymptotic series expansions for  $z \rightarrow z_0$  are characterised by possessing different asymptotic series for different sectors of the  $z$ -plane around  $z = z_0$ . Thus, for instance,  $g(z) := \exp(z) + \exp(-z) \tanh(1/z)$  has the following asymptotic forms for  $z \rightarrow 0$ :  $g(z) \sim 2 \cosh(z) \sim 2 + z^2 + \dots$ , when  $\text{Re}(z) > 0$ , and  $g(z) \sim 2 \sinh(z) \sim 2z + z^3/3 + \dots$ , when  $\text{Re}(z) < 0$  (Lauwerier 1977, p. 11). The fact that in different sectors around a point in the complex  $z$ -plane a function can have different asymptotic expansions with respect to the same asymptotic sequence, is referred to as the *Stokes phenomenon* (Watson 1952, Dingle 1973); ‘Stokes lines’ are thus the branch cuts in our discussions.

The Taylor series of analytic functions based on point  $z_0$  are *convergent* uniform asymptotic series for  $z \rightarrow z_0$  (similarly for the Laurent series). It can be shown (Lauwerier 1977, pp. 12-14) that an analytic function can be associated with even a divergent asymptotic series corresponding to the asymptotic sequence  $\{(z - z_0)^n | n = 0, 1, \dots\}$ ; the given (divergent) series is its asymptotic expansion. Thus, for instance, after Borel (or Euler) transformation (Whittaker and Watson 1927, pp. 154 and 155, Dingle 1973, pp. 405-408, Lauwerier 1977, pp. 45-50) of the divergent series  $f(z) := \sum_{n=0}^{\infty} (-1)^n n! z^n$ , one formally obtains  $f_B(z) := \sum_{n=0}^{\infty} (-1)^n z^n = (1+z)^{-1}$ . Through the Borel back-transformation of  $f_B(z)$ ,  $\bar{f}_B(z) := \int_0^{\infty} dx \exp(-x) f_B(xz)$ , one obtains a function, i.e.  $\bar{f}_B(z)$ , which is analytic in the sector  $-\pi < \arg(z) < \pi$  of the  $z$ -plane.<sup>55</sup> Through replacing  $f_B(xz)$  by its *formal* geometric expansion  $1 - xz + (xz)^2 - \dots$ , the term-by-term evaluation of the latter integral yields the original divergent series. We observe that the divergence of the original asymptotic series is closely associated with the restricted sector of the  $z$ -plane around  $z = 0$  over which  $\bar{f}_B(z)$  is analytic.

### 3. Many-valued functions: Physical and non-physical Riemann sheets

Many-valuedness (Whittaker and Watson 1927, pp. 96-98, Titchmarsh 1939, pp. 138-164) is a generic property of correlation functions of complex energy pertaining to systems in the thermodynamic limit (see § A.4). In particular, the single-particle Green function and the self-energy operator are many-valued (see § II).

An  $n$ -valued function of complex variable  $z$  over domain  $\mathcal{D}$  may be thought of as embodying  $n$  branches of a single-valued function over the extended domain consisting of the union of  $n$  replicas of  $\mathcal{D}$ ; since these domains signify the same region of the complex plane, they are to be distinguished by considering them as being located on different sheets, namely the *Riemann sheets* (RSs), of the complex plane. In the present paper we single out one particular branch of the many-valued functions that we encounter, such as the single-particle Green function, because of its *physical* significance. To this particular branch we refer as the function in question on the *physical* RS or the ‘physical’ function (compare with  $G(\varepsilon)$  in Eq. (1) which we have designated as the ‘physical’ single-particle Green function).

Throughout this paper we employ the following notational conventions. Let  $f(\varepsilon)$  be a function of real energy variable  $\varepsilon$ , which we consider to describe some dynamical physical property of the many-particle system under consideration. By  $\tilde{f}(z)$  we denote the analytic function that *uniformly* approaches  $f(\varepsilon)$  when  $z \rightarrow \varepsilon$ ; the so-called ‘edge-of-the-wedge’ theorem (Streater and Wightman 1964) asserts uniqueness of  $\tilde{f}(z)$ .<sup>56</sup> We refer to  $\tilde{f}(z)$  as  $\tilde{\tilde{f}}(z)$  on the physical RS;  $\tilde{\tilde{f}}(z)$  denotes the above-mentioned single-valued function over the union of all the RSs. We recall that in the main text we have referred to, for example,  $\tilde{G}(z)$  in Eq. (4) as the single-particle Green function on the physical RS.

An example will clarify this. Let  $f(\varepsilon) := \ln(\varepsilon)$ , for  $\varepsilon > 0$ . We can identify  $\ln(z)$  by  $\tilde{f}(z)$  since for  $z \rightarrow \varepsilon$ , with  $\varepsilon > 0$ , indeed  $\ln(z) \rightarrow \ln(\varepsilon)$ . One can easily verify that  $\text{Ln}_n(z) := \ln|z| + i\{\arg(z) + 2\pi n\}$ , with  $n$  an integer (positive, zero

<sup>55</sup> It can be shown that  $\bar{f}_B(z) \equiv \exp(1/z) E_1(1/z)/z$  where  $E_1(z)$  stands for the exponential-integral function (see, e.g., Abramowitz and Stegun 1972, p. 228). We note that  $\exp(1/z)$  has an isolated essential singularity at  $z = 0$  (see § A.1) and that  $E_1(1/z)$  has branch points at  $z = 0$  and  $1/z = 0$ .

<sup>56</sup> The condition of *uniformity* can be shown, *a posteriori*, to be redundant: the existence of the limit  $\tilde{f}(z) \rightarrow f(\varepsilon)$ , as  $z \rightarrow \varepsilon$ , with  $\varepsilon$  varying over an *open* interval where  $f(\varepsilon)$  is *continuous*, implies uniformity of  $\tilde{f}(z) \rightarrow f(\varepsilon)$ ; see p. 75 in Streater and Wightman (1964).

and negative) and  $-\pi \leq \arg(z) < \pi$ , is *the* analytic function (i.e.  $\tilde{f}(z)$ ) of which  $\tilde{f}(z) := \ln(z)$  is the physical branch; thus this branch corresponds to  $n = 0$ . The physical branch  $\tilde{f}(z)$ , like any other branch of  $\tilde{f}(z)$ , is a many-valued function (branch points are *not* removable [see § A.1] singularities). It has two branch points, one at  $z = 0$  and the other at  $1/z = 0$ . Our above convention  $-\pi \leq \arg(z) < \pi$  specifies the branch cut of  $\tilde{f}(z)$ , which connects the two branch points of  $\tilde{f}(z)$ , to be along the negative  $\varepsilon$ -axis. One can determine *all* branches of  $\tilde{f}(z)$  from the knowledge of any of its branches. Two of infinitely many branches of  $\tilde{f}(z)$ ,  $\tilde{f}_1(z)$  and  $\tilde{f}_2(z)$  say, that can be obtained by means of a *direct* analytic continuation of  $\tilde{f}(z)$ , through its branch cut, into two different non-physical RSs are *uniquely* determined from the following requirements:  $\lim_{\eta \downarrow 0} \{\tilde{f}_1(\varepsilon + i\eta) - \tilde{f}(\varepsilon - i\eta)\} = 0$  and  $\lim_{\eta \downarrow 0} \{\tilde{f}_2(\varepsilon - i\eta) - \tilde{f}(\varepsilon + i\eta)\} = 0$ , for  $\varepsilon < 0$ . It is not difficult to verify that  $\tilde{f}_1(z) \equiv \text{Ln}_{-1}(z)$  and  $\tilde{f}_2(z) \equiv \text{Ln}_{+1}(z)$ .

#### 4. A physically-motivated example

Here we apply the above concepts to a simple model function which accommodates a number of salient features of the physical functions which we deal with in the main part of this paper. We shall particularly emphasise the role played by the process of evaluating the thermodynamic limit in modifying the nature of the singular points of dynamic correlation functions.

Consider  $f(\varepsilon; N, \Omega) := \frac{1}{\Omega} \sum_{\ell} [\Theta(\varepsilon_{\ell} - e_0) - \Theta(\varepsilon_{\ell} - e_1)] / (\varepsilon_{\ell} - \varepsilon)$ , with  $\varepsilon_{\ell+1} > \varepsilon_{\ell}$ . Here  $e_0$  and  $e_1$  are finite constants for which we assume  $e_0 < \varepsilon_{\ell} < e_1$  for *some* values of  $\ell$ ;  $N$  and  $\Omega$  indicate that  $f$  is a function of the number  $N$  of particles as well as the volume  $\Omega$  of the system. In the ‘thermodynamic limit’ ( $N \rightarrow \infty$ ,  $\Omega \rightarrow \infty$  and finite concentration  $C := N/\Omega$ ),  $f$  is only a function of  $C$ . A brief glance at the contents of § II should clarify our present choice for  $f$ .

Suppose that in the thermodynamic limit  $(\varepsilon_{\ell+M} - \varepsilon_{\ell}) \rightarrow 0$  for *any* finite value of  $M$ . This condition implies that in taking the limit, the function  $f(\varepsilon; N, \Omega)$  becomes ill-defined for *any* real  $\varepsilon$  in the interval  $[e_0, e_1]$ ; an  $\varepsilon$  in this interval will be ‘pinched’ (Itzykson and Zuber 1988, pp. 302 and 303) by poles of  $f(\varepsilon; N, \Omega)$ . To avoid this problem, the thermodynamic limit has to be effected *after* replacing the real energy variable  $\varepsilon$  by a complex energy variable  $z$ , since a complex  $z$  cannot be ‘pinched’ by the real poles of  $f(z; N, \Omega)$  as  $N$  and  $\Omega$  approach infinity.

To be specific, let us now assume that in the thermodynamic limit poles of  $f(z; N, \Omega)$  populate the interval  $[e_0, e_1]$  with a constant density equal to  $A \times \Omega$ . We write

$$\begin{aligned} f(z; N, \Omega) &= \frac{1}{\Omega} \sum_{\ell} \frac{\Theta(\varepsilon_{\ell} - e_0) - \Theta(\varepsilon_{\ell} - e_1)}{\varepsilon_{\ell} - z} \\ &\cong A \int_{e_0}^{e_1} \frac{d\varepsilon'}{\varepsilon' - z} =: \tilde{f}(z; C), \quad \text{for } N \rightarrow \infty, \Omega \rightarrow \infty. \end{aligned} \quad (\text{A1})$$

One trivially obtains

$$\tilde{f}(z; C) = A \{\ln(z - e_1) - \ln(z - e_0)\}. \quad (\text{A2})$$

Evidently, the process of evaluating the thermodynamic limit has led to a dramatic change in  $f(z; N, \Omega)$ , transforming it into  $\tilde{f}(z; C)$  which analytically is distinct from  $f(z; N, \Omega)$ . For instance,  $\tilde{f}(z; C)$  has *no* poles, rather it possesses two infinite-order branch points, at  $z = e_0$  and  $z = e_1$ . Consequently, contrary to  $f(z; N, \Omega)$ , which is single-valued,  $\tilde{f}(z; C)$  is a many-valued function of  $z$ .

We shall now explicitly demonstrate that  $\tilde{f}(z; C)$  as presented in Eq. (A2) is the ‘physical’ branch of  $\tilde{f}(z)$  (see § A.3). To this end, consider  $\tilde{f}(\varepsilon \pm i\eta; C) \equiv A \int_{e_0}^{e_1} d\varepsilon' / (\varepsilon' - \varepsilon \mp i\eta)$ . For  $\eta \downarrow 0$  we have  $1/(\varepsilon' - \varepsilon \mp i\eta) = \mathcal{P}\{1/(\varepsilon' - \varepsilon)\} \pm i\pi\delta(\varepsilon' - \varepsilon)$ , where  $\mathcal{P}$  denotes the Cauchy ‘principal value’. It is then trivially seen that  $\tilde{f}(\varepsilon \pm i\eta; C) = A\{\ln|e_1 - \varepsilon| - \ln|e_0 - \varepsilon|\} \pm i\pi A\Theta(\varepsilon - e_0)\Theta(e_1 - \varepsilon)$  for  $\eta \downarrow 0$ . Hence, for  $\varepsilon < e_0$  and  $\varepsilon > e_1$  the ‘physical’ branch  $\tilde{f}(\varepsilon \pm i\eta; C)$  must be real-valued, while for  $\varepsilon \in (e_0, e_1)$  it must satisfy  $\lim_{\eta \downarrow 0} \{\tilde{f}(\varepsilon + i\eta; C) - \tilde{f}(\varepsilon - i\eta; C)\} = 2\pi i A$ ; thus the interval  $[e_0, e_1]$  constitutes the branch cut of  $\tilde{f}(z; C)$ . These conditions are *exactly* fulfilled by the expression in Eq. (A2); note that  $\ln(z)$ , as distinct from  $\text{Ln}_n(z)$  for  $n \neq 0$ , stands for the principal-value logarithm (see, e.g., Abramowitz and Stegun 1972, p. 67). Thus  $\tilde{f}(z; C)$  in Eq. (A2) is indeed the analytic continuation of  $f(\varepsilon; C)$  into the *physical* RS.

It is instructive to consider two examples concerning continuations of  $\tilde{f}(z; C)$  into non-physical RSs. Let  $\tilde{g}(z) := A\{\text{Ln}_1(z - e_1) - \ln(z - e_0)\}$ . It is easily verified that for  $e_0 < \varepsilon < e_1$ ,  $\lim_{\eta \downarrow 0} \{\tilde{f}(\varepsilon + i\eta; C) - \tilde{g}(\varepsilon - i\eta)\} = 0$ , which implies that  $\tilde{g}(z)$  is *the* analytic continuation of  $\tilde{f}(z; C)$  from the upper-half plane of the physical RS through

the branch cut  $[e_0, e_1]$  into the lower-half plane of a non-physical RS. That is  $\tilde{g}(z)$  is a branch of  $\tilde{f}(z; C)$  on a non-physical RS. As for the second example, consider  $\tilde{h}(z) := A\{\text{Ln}_{-1}(z - e_1) - \ln(z - e_0)\}$ . For  $\varepsilon \in (e_0, e_1)$  we have  $\lim_{\eta \downarrow 0} \{\tilde{f}(\varepsilon - i\eta; C) - \tilde{h}(\varepsilon + i\eta)\} = 0$  so that  $\tilde{h}(z)$  is the analytic continuation of  $\tilde{f}(z; C)$  from the lower-half plane through the branch cut  $[e_0, e_1]$  into the upper-half plane of a non-physical RS.

## APPENDIX B: ASYMPTOTIC BEHAVIOUR OF THE SELF-ENERGY AT LARGE ENERGIES

Here we demonstrate that, for  $|z| \rightarrow \infty$ ,  $\tilde{\Sigma}(z) \sim \Sigma^{HF}$ , the Hartree-Fock self-energy,

$$\Sigma^{HF}(\mathbf{r}, \mathbf{r}') \equiv \Sigma^H(\mathbf{r}, \mathbf{r}') + \Sigma^F(\mathbf{r}, \mathbf{r}') \equiv \frac{1}{\hbar} v_H(\mathbf{r}; [n]) \delta(\mathbf{r} - \mathbf{r}') - \frac{1}{2\hbar} v(\mathbf{r} - \mathbf{r}') \varrho(\mathbf{r}, \mathbf{r}'), \quad (\text{B1})$$

where  $v_H(\mathbf{r}; [n]) := \int d^d r' v(\mathbf{r} - \mathbf{r}') n(\mathbf{r}')$  stands for the Hartree potential, which is a functional of the electronic number density  $n$  in the ground state, and  $\varrho$  for the reduced single-particle density matrix. We note that  $\varrho$  in the definition for  $\Sigma^F$  pertains to the fully interacting system and is distinct from the Slater-Fock reduced density matrix  $\varrho_0$ ; contrary to the former, the latter is idempotent, i.e.  $\varrho_0 \varrho_0 \equiv \varrho_0$ , whereas  $\varrho \varrho \neq \varrho$ . This difference may have some far-reaching consequences. For instance, in a uniform isotropic system of electrons interacting through the long-range Coulomb interaction function, for  $\Sigma^F(k)$  evaluated in terms of  $\varrho_0$ , which we denote by  $\Sigma_s^F(k)$ , one has (Ashcroft and Mermin 1981, p. 334) <sup>57</sup>

$$\Sigma_s^F(k) = -\frac{2e^2}{\pi\hbar} k_F \mathcal{F}(k/k_F), \quad \mathcal{F}(x) := \frac{1}{2} + \frac{1-x^2}{4x} \ln \left| \frac{1+x}{1-x} \right|; \quad (\text{B2})$$

the first derivative with respect to  $k$  of  $\Sigma_s^F(k)$  is seen to be logarithmically divergent at  $k = k_F$ , an aspect which *may* be an artifact of  $\Sigma_s^F(k)$  and may not be shared by  $\Sigma^F(k)$ .

From the Dyson equation we have  $\Sigma(\varepsilon) = G_0^{-1}(\varepsilon) - G^{-1}(\varepsilon)$ . Therefore to deduce the asymptotic expansion of  $\Sigma(\varepsilon)$  for  $|\varepsilon| \rightarrow \infty$ , we need to determine those for  $G_0^{-1}(\varepsilon)$  and  $G^{-1}(\varepsilon)$ . The asymptotic series for  $G_0^{-1}(\varepsilon)$  and  $G^{-1}(\varepsilon)$  can be deduced from those pertaining to  $G_0(\varepsilon)$  and  $G(\varepsilon)$ , respectively, through reliance on the following result from the theory of asymptotic analysis (see Copson 1965, pp. 8 and 9): Let  $\tilde{f}(z)$  have the following asymptotic expansion for  $|z| \rightarrow \infty$ , where  $z$  is a complex variable:  $\tilde{f}(z) \sim f_0 + f_1/z + f_2/z^2 + \dots$ , with  $f_0, f_1, \dots$  constants, independent of  $z$ . Then, provided that  $f_0 \neq 0$ ,  $1/\tilde{f}(z) \sim 1/f_0 + \tilde{f}_1/z + \tilde{f}_2/z^2 + \dots$  holds for  $|z| \rightarrow \infty$ , where  $\tilde{f}_1 = -f_1/f_0^2$ ,  $\tilde{f}_2 = (f_1^2 - f_0 f_2)/f_0^3$ , etc. <sup>58</sup>

From the Lehmann representation for  $G(\varepsilon)$  in Eq. (1) it follows that

$$G(\varepsilon) \sim \frac{G_{\infty 1}}{\varepsilon} + \frac{G_{\infty 2}}{\varepsilon^2} + \dots, \quad \text{for } |\varepsilon| \rightarrow \infty, \quad (\text{B3})$$

where

$$G_{\infty 1}(\mathbf{r}, \mathbf{r}') = \hbar \sum_s \Lambda_s(\mathbf{r}) \Lambda_s^*(\mathbf{r}') \equiv \hbar \delta(\mathbf{r} - \mathbf{r}'), \quad (\text{B4})$$

$$G_{\infty 2}(\mathbf{r}, \mathbf{r}') = \hbar \sum_s \varepsilon_s \Lambda_s(\mathbf{r}) \Lambda_s^*(\mathbf{r}') \equiv \hbar \left( \Xi_{<}(\mathbf{r}, \mathbf{r}') + \Xi_{>}(\mathbf{r}, \mathbf{r}') \right); \quad (\text{B5})$$

for  $\Xi_{<}$  and  $\Xi_{>}$  see Eqs. (B9) and (B10) below. A similar expression to Eq. (B3) holds for  $G_0(\varepsilon)$ . Therefore from our above considerations with regard to the asymptotic series for  $1/\tilde{f}(z)$  in terms of the coefficients of that for  $\tilde{f}(z)$ , it follows that  $\Sigma(\varepsilon)$  possesses the following asymptotic series expansion

$$\Sigma(\varepsilon) \sim \Sigma_{\infty 0} + \frac{\Sigma_{\infty 1}}{\varepsilon} + \frac{\Sigma_{\infty 2}}{\varepsilon^2} + \dots, \quad \text{for } |\varepsilon| \rightarrow \infty. \quad (\text{B6})$$

<sup>57</sup> For this system,  $\Sigma^H(k)$  is divergent but cancels an equally-divergent contribution due to the field of the positively-charged uniform background.

<sup>58</sup> When  $f_0 = 0$  and  $f_1 \neq 0$ , then one should apply this result to  $\tilde{g}(z) := z\tilde{f}(z)$ ; from the asymptotic series for  $1/\tilde{g}(z)$  that for  $1/\tilde{f}(z)$  is obtained through a multiplication by  $z$ .



Here we are interested in the leading asymptotic term  $\Sigma_{\infty_0}$  which our above considerations indicate to be determined from  $G_{0;\infty_2}$  and  $G_{\infty_2}$  as follows

$$\Sigma_{\infty_0} = \frac{1}{\hbar^2} \{ -G_{0;\infty_2} + G_{\infty_2} \}. \quad (\text{B7})$$

For the many-body Hamiltonian of the form

$$\hat{H} = \int d^d r \, \hat{\psi}^\dagger(\mathbf{r}) \left[ \frac{-\hbar^2}{2m_e} \nabla^2 + u(\mathbf{r}) \right] \hat{\psi}(\mathbf{r}) + \frac{1}{2} \int d^d r d^d r' \, \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{r}') v(\mathbf{r} - \mathbf{r}') \hat{\psi}(\mathbf{r}') \hat{\psi}(\mathbf{r}), \quad (\text{B8})$$

with  $u(\mathbf{r})$  the local external potential, making use of the definition for the Lehmann amplitudes and energies as given in Eq. (3) and the fact that  $E_{M,s} |\Psi_{M,s}\rangle = \hat{H} |\Psi_{M,s}\rangle$ , one readily obtains

$$\begin{aligned} \Xi_{<}(\mathbf{r}, \mathbf{r}') &:= \sum_s \theta(\mu - \varepsilon_s) \varepsilon_s \Lambda_s(\mathbf{r}) \Lambda_s^*(\mathbf{r}') \\ &= -\langle \Psi_{N,0} | \hat{\psi}^\dagger(\mathbf{r}') \left[ \hat{H}, \hat{\psi}(\mathbf{r}) \right]_- | \Psi_{N,0} \rangle, \end{aligned} \quad (\text{B9})$$

$$\begin{aligned} \Xi_{>}(\mathbf{r}, \mathbf{r}') &:= \sum_s \theta(\varepsilon_s - \mu) \varepsilon_s \Lambda_s(\mathbf{r}) \Lambda_s^*(\mathbf{r}') \\ &= -\langle \Psi_{N,0} | \left[ \hat{H}, \hat{\psi}(\mathbf{r}) \right]_- \hat{\psi}^\dagger(\mathbf{r}') | \Psi_{N,0} \rangle. \end{aligned} \quad (\text{B10})$$

By applying the canonical anti-commutation relations for the field operators in the Schrödinger picture one arrives at the following result

$$G_{\infty_2}(\mathbf{r}, \mathbf{r}') = \hbar \left\{ \left[ \frac{-\hbar^2}{2m_e} \nabla^2 + u(\mathbf{r}) + v_H(\mathbf{r}; [n]) \right] \delta(\mathbf{r} - \mathbf{r}') - \frac{1}{2} v(\mathbf{r} - \mathbf{r}') \varrho(\mathbf{r}, \mathbf{r}') \right\}. \quad (\text{B11})$$

The corresponding expression for the non-interacting Green function follows from this expression by setting the coupling constant of the electron-electron interaction equal to zero,

$$G_{0;\infty_2}(\mathbf{r}, \mathbf{r}') = \hbar \left[ \frac{-\hbar^2}{2m_e} \nabla^2 + u(\mathbf{r}) \right] \delta(\mathbf{r} - \mathbf{r}'). \quad (\text{B12})$$

From Eqs. (B7), (B11), (B12) and (B1) we obtain the following result

$$\Sigma_{\infty_0} \equiv \Sigma^{HF}, \quad (\text{B13})$$

which completes our demonstration. We note that Eqs. (B3) and (B6) hold equally for  $\tilde{G}(z)$  and  $\tilde{\Sigma}(z)$ , respectively, with merely  $\varepsilon$  replaced by  $z$ .

### APPENDIX C: THE KRAMERS-KRONIG RELATIONS FOR THE SELF-ENERGY; APPLICATIONS TO ISOTROPIC FERMI AND MARGINAL-FERMI LIQUIDS

The analyticity of a *function* of the complex variable  $z = x + iy$  in a domain  $\mathcal{D}$  of the  $z$ -plane implies that in  $\mathcal{D}$  its real (imaginary) part is up to a constant *uniquely* determined by its imaginary (real) part (Morse and Feshbach 1953, pp. 356-358). For functions, such as  $\tilde{f}(z)$ , whose singularities are located along the  $x$ -axis and whose real and imaginary parts,  $u(x, y)$  and  $v(x, y)$ , approach zero not slower than  $1/\varrho^p$ , with  $p > 0$ , as  $\varrho := (x^2 + y^2)^{1/2} \rightarrow \infty$ , the Kramers-Kronig relations establish the mentioned unique relationships between  $u(x, 0^\pm)$  and  $v(x, 0^\pm)$ .

The self-energy  $\tilde{\Sigma}(z)$  is analytic over the entire  $z$ -plane with the exception of the real axis.<sup>59</sup> Two aspects should be addressed before a Kramers-Kronig pair of relations can be set up for the self-energy. First, as the considerations in Appendix B have demonstrated,  $\tilde{\Sigma}(z) \sim \Sigma^{HF}$  for  $|z| \rightarrow \infty$ , so that we need first to introduce

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<sup>59</sup> This follows from the Dyson equation  $\tilde{\Sigma}(z) = \tilde{G}_0^{-1}(z) - \tilde{G}^{-1}(z)$  and the fact that neither  $\tilde{G}_0(z)$  nor  $\tilde{G}(z)$  possesses zero eigenvalues for  $\text{Im}(z) \neq 0$  (see last paragraph in § II).

$$\tilde{\Sigma}_*(z) := \tilde{\Sigma}(z) - \Sigma^{HF}, \quad (C1)$$

which, following Eq. (B6), approaches zero according to  $\sim \Sigma_{\infty_1}/z$ . Second, as  $\tilde{\Sigma}(z)$  is an operator, it cannot be assigned real and imaginary parts; moreover, as  $\tilde{\Sigma}(z)$  is not gauge invariant, real and imaginary parts of the matrix elements  $\{\langle \alpha | \tilde{\Sigma}(z) | \beta \rangle\}$ , for a given representation, are dependent upon the choice of the gauge. We need therefore generalise for operators the notions ‘real part’ and ‘imaginary part’ before the corresponding Kramers-Kronig relations can be meaningfully defined. We introduce

$$\tilde{\Sigma}'(z) := \frac{1}{2} \{ \tilde{\Sigma}(z) + \tilde{\Sigma}^\dagger(z) \}, \quad \tilde{\Sigma}''(z) := \frac{1}{2i} \{ \tilde{\Sigma}(z) - \tilde{\Sigma}^\dagger(z) \}. \quad (C2)$$

We similarly introduce  $\tilde{\Sigma}'_*(z)$  and  $\tilde{\Sigma}''_*(z)$ ;  $\tilde{\Sigma}'(z)$  and  $i\tilde{\Sigma}''(z)$  ( $\tilde{\Sigma}'_*(z)$  and  $i\tilde{\Sigma}''_*(z)$ ) are Hermitian and anti-Hermitian components of  $\tilde{\Sigma}(z)$  ( $\tilde{\Sigma}_*(z)$ ) and are unique. This *uniqueness* enables us to deduce from

$$\tilde{\Sigma}_*(\varepsilon \pm i\eta) = \pm \frac{1}{\pi i} \mathcal{P} \int_{-\infty}^{\infty} d\varepsilon' \frac{\tilde{\Sigma}_*(\varepsilon' \pm i\eta)}{\varepsilon' - \varepsilon}, \quad (C3)$$

which follows from the application of the Cauchy theorem together with the above-indicated analytic and asymptotic properties of  $\tilde{\Sigma}_*(z)$ , the following pair of representation-free Kramers-Kronig relations

$$\Sigma'_*(\varepsilon) = -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} d\varepsilon' \frac{\text{sgn}(\mu - \varepsilon') \Sigma''_*(\varepsilon')}{\varepsilon' - \varepsilon}, \quad (C4)$$

$$\Sigma''_*(\varepsilon) = +\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} d\varepsilon' \frac{\text{sgn}(\mu - \varepsilon) \Sigma'_*(\varepsilon')}{\varepsilon' - \varepsilon}. \quad (C5)$$

In obtaining these relations we have made use of our convention in Eq. (5). We note that  $\Sigma'(\varepsilon) \equiv \Sigma^{HF} + \Sigma'_*(\varepsilon)$  and  $\Sigma''(\varepsilon) \equiv \Sigma''_*(\varepsilon)$ .

## 1. Fermi liquids

Now we employ Eq. (C4) and obtain from <sup>60</sup>  $\Sigma''(k; \varepsilon) \equiv \text{Im} \Sigma(k; \varepsilon) \sim \mp \alpha_k (\varepsilon - \varepsilon_F)^2$ ,  $\varepsilon \gtrless \varepsilon_F$ , the associated  $\Sigma'(k; \varepsilon)$ ; in this way we obtain Eq. (29) as well as an explicit expression for  $\beta_k$  in terms of  $\alpha_k$  and  $\Sigma''(k; \varepsilon)$ . To this end, we subdivide the interval of the  $\varepsilon'$ -integration in Eq. (C4) into three subintervals,  $(-\infty, -\Delta]$ ,  $(-\Delta, +\Delta]$  and  $(+\Delta, +\infty)$ , where we assume  $\Delta > |\delta\varepsilon|$  with  $\delta\varepsilon := \varepsilon - \varepsilon_F$ . Upon change of variables and transforming integrals over the negative  $\varepsilon'$ -axis into those over the positive  $\varepsilon'$ -axis we obtain

$$\Sigma'_*(k; \varepsilon) \sim \frac{-2\alpha_k \delta\varepsilon}{\pi} \mathcal{P} \int_0^\Delta d\varepsilon' \frac{\varepsilon'^2}{\varepsilon'^2 - \delta\varepsilon^2} + \frac{1}{\pi} \int_\Delta^\infty d\varepsilon' \left\{ \frac{\Sigma''(k; \varepsilon' + \varepsilon_F)}{\varepsilon' - \delta\varepsilon} + \frac{\Sigma''(k; -\varepsilon' + \varepsilon_F)}{\varepsilon' + \delta\varepsilon} \right\}. \quad (C6)$$

The first integral on the right-hand side of Eq. (C6) is standard (Gradshteyn and Ryzhik 1965, p. 59) and one has  $\mathcal{P} \int_0^\Delta d\varepsilon' \frac{\varepsilon'^2}{\varepsilon'^2 - \delta\varepsilon^2} = \Delta - (\delta\varepsilon/2) \ln |(\Delta + \delta\varepsilon)/(\Delta - \delta\varepsilon)|$ . Since, in the second integral on the right-hand side of Eq. (C6),  $\varepsilon' \geq \Delta$  and since  $\Delta > |\delta\varepsilon|$ , we replace  $1/(\varepsilon' - \delta\varepsilon)$  and  $1/(\varepsilon' + \delta\varepsilon)$  by their respective geometric series expansions, in powers of  $\delta\varepsilon/\varepsilon'$ , and thus obtain a powers series in  $\delta\varepsilon$  for the second integral. Collecting terms of the zeroth and the first order in  $\delta\varepsilon$ , while making use of Eq. (29), we obtain

$$\Sigma(k; \varepsilon_F) \equiv \Sigma'(k; \varepsilon_F) \cong \Sigma^{HF}(k) + S_0^{(\Delta)}(k), \quad \beta_k \cong S_1^{(\Delta)}(k) - \frac{2\Delta\alpha_k}{\pi}, \quad (C7)$$

where

$$S_m^{(\Delta)}(k) := \frac{1}{\pi} \int_\Delta^\infty \frac{d\varepsilon'}{\varepsilon'^{m+1}} \left\{ \Sigma''(k; \varepsilon' + \varepsilon_F) + (-1)^m \Sigma''(k; -\varepsilon' + \varepsilon_F) \right\}. \quad (C8)$$

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<sup>60</sup> Note that  $\varepsilon_F \equiv \mu_N$ ,  $\alpha_k \geq 0$  and that this asymptotic relation is valid for  $k$  in a neighbourhood of  $k_F$ , say for  $0 \leq k \lesssim 3k_F$  — see footnote 26.

In Eq. (C7), ‘ $\cong$ ’ signifies the approximate nature of the results in so far as  $\Delta$  is finite and, moreover, in evaluating the  $\varepsilon'$ -integral over  $(-\Delta, +\Delta]$  we have employed a truncated asymptotic expansion for  $\Sigma''(k; \varepsilon)$ . As should be evident, our choice for a finite  $\Delta$  has its root in the requirement  $\Delta > |\delta\varepsilon|$ . Since Eqs. (C7) and (C8) are valid independently of the value of  $\delta\varepsilon$ , it is possible to let  $\Delta$  in these equations approach zero; because  $\Sigma''(k; \pm\varepsilon' + \varepsilon_F) \sim \varepsilon'^2$  for  $\varepsilon' \rightarrow 0$ ,  $S_m^{(\Delta, 0)}(k)$ , which involves  $1/\varepsilon'^{m+1}$ , exists for  $m = 0, 1$ , and is equal to  $S_m^{(0)}(k)$ ; on the other hand, for  $m \geq 2$ ,  $S_m^{(\Delta)}(k)$  does *not* have a limit for  $\Delta \downarrow 0$ . Thus we can write

$$\Sigma(k; \varepsilon_F) \equiv \Sigma'(k; \varepsilon_F) = \Sigma^{HF}(k) + S_0^{(0)}(k), \quad \beta_k = S_1^{(0)}(k). \quad (C9)$$

We point out that, since, for  $\varepsilon' \geq 0$ ,  $\Sigma''(k; \varepsilon' + \varepsilon_F) \leq 0$  and  $\Sigma''(k; -\varepsilon' + \varepsilon_F) \geq 0$ ,<sup>61</sup> it follows that  $S_m^{(\Delta)}(k) \leq 0$  for all odd values of  $m$ . Specifically, following Eq. (C9), we have  $\beta_k \leq 0$  (concerning the range of  $k$  over which this result can be relied upon, see footnotes 26 and 60). As we have indicated in § V (see text following Eq. (29)), for  $k \rightarrow k_F$  this inequality guarantees  $Z_{k_F} \leq 1$ .

It is interesting to note that  $S_m^{(\Delta)}(k)$ ,  $m = 0, 1$ , can be written in the following appealing form

$$S_0^{(\Delta)}(k) := \frac{1}{\pi} \operatorname{Im} \left[ \int_{\mu+\Delta}^{i\infty} \frac{dz}{(z-\mu)} \{ \tilde{\Sigma}(k; z) + \tilde{\Sigma}(k; 2\mu - z) - 2\Sigma^{HF}(k) \} \right], \quad (C10)$$

$$S_1^{(\Delta)}(k) := \frac{1}{\pi} \operatorname{Im} \left[ \int_{\mu+\Delta}^{i\infty} \frac{dz}{(z-\mu)^2} \{ \tilde{\Sigma}(k; z) - \tilde{\Sigma}(k; 2\mu - z) \} \right], \quad (C11)$$

where the term  $2\Sigma^{HF}(k)$ , which is real-valued, has been introduced in order to render the corresponding integral existent (see Eq. (B13)). In Eqs. (C10) and (C11) we have further restored  $\mu$  which originates from Eq. (C4) and which is *not* exactly equal to  $\varepsilon_F$  (see § IV.A). We point out that in these expressions, contrary to that in Eq. (C8),  $\Delta$  *cannot* be set equal to zero, since for  $\Delta = 0$  the integrals enclosed by square brackets (logarithmically) diverge as a consequence of the integrands being proportional to  $1/(z-\mu)$  for  $z \rightarrow \mu$ ; since, however, the singular contributions are real-valued,  $\Delta$  can be made as small as desired, provided that it is kept non-zero; thus  $S_m^{(0+)}(k)$ ,  $m = 0, 1$ , is well-defined.

The relevance of Eqs. (C10) and (C11) rests in the following: The fact that  $\tilde{\Sigma}(k; z)$  is *analytic* over the entire  $z$ -plane (excluding the real axis outside  $(\mu_N, \mu_{N+1})$ ), implies that if  $\partial^\ell S_m^{(0+)}(k)/\partial k^\ell$ ,  $m = 0, 1$  (including  $\ell = 0$ ), is singular at some  $k$ , this singularity must be shared by the integrand of  $\partial^\ell S_m^{(0+)}(k)/\partial k^\ell$ ,  $m = 0, 1$ , for *all*  $z$ ,  $\operatorname{Im}(z) \neq 0$ . In other words, Eqs. (C10) and (C11) make explicit that possible singularities in  $\partial^\ell S_m^{(0+)}(k)/\partial k^\ell$ ,  $m = 0, 1$ , are *not* specific to  $\tilde{\Sigma}(k; z)$  for some  $z = z_0$ , but to  $\tilde{\Sigma}(k; z)$  for *all*  $z$ ,  $\operatorname{Im}(z) \neq 0$ .<sup>62</sup>

In order to demonstrate the utility of our above considerations, let us assume  $\Sigma^{HF}(k)$ , similar to  $\Sigma_s^{HF}(k)$  (see Eq. (B2)), has a divergent first-order derivative with respect to  $k$  at  $k = k_F$  (see Appendix B). For the system under consideration to be a Fermi liquid,  $\Sigma(k; \varepsilon_F)$  must, by definition, be a continuously differentiable function of  $k$  at  $k = k_F$  (see §§ I and IV.C). From Eq. (C9) it follows that the assumed singularity due to  $\partial \Sigma^{HF}(k)/\partial k$  at

<sup>61</sup> Violation of these inequalities implies breakdown of causality and instability of the ground state of the system (see Eq. (26) in comparison with Eq. (1)).

<sup>62</sup> Consider  $g(k) := \int_{\mathcal{C}} dz \tilde{f}(k; z)$ , where the contour of integration  $\mathcal{C}$  lies inside an open domain  $\mathcal{D}$  of the complex  $z$ -plane where  $\tilde{f}(k; z)$  is *analytic*. Suppose that the  $\ell$ th-order derivative (including  $\ell = 0$ ) with respect to  $k$  of  $g(k)$  is divergent at  $k = k_0$ . This divergence *cannot* be due to some *isolated* singularity (singularities) of  $\partial^\ell \tilde{f}(k; z)/\partial k^\ell|_{k=k_0}$  along  $\mathcal{C}$ , for *analyticity* of  $\tilde{f}(k; z)$  enables one to deform  $\mathcal{C}$  inside  $\mathcal{D}$ , thus avoiding the mentioned singularity (singularities), without changing the value of the integral (Cauchy’s theorem); exceptions to this concern the cases where  $\partial^\ell \tilde{f}(k; z)/\partial k^\ell|_{k=k_0}$  is divergent either at the end-points of  $\mathcal{C}$ , which cannot be dislocated, or at two neighbouring points which ‘pinch’  $\mathcal{C}$ . Note that in Eqs. (C10) and (C11) the contours of integration are arbitrary, as long as they connect  $\mu + \Delta$  with the point of infinity (the ‘end-points’) and are located on the upper-half of the  $z$ -plane (signifying  $\mathcal{D}$ ). Therefore, excluding end-point and ‘pinch’ singularities, divergence of  $\partial^\ell g(k)/\partial k^\ell$  at  $k = k_0$  implies divergence of  $\partial^\ell \tilde{f}(k; z)/\partial k^\ell$  at  $k = k_0$  for *all*  $z$  inside  $\mathcal{D}$ . In Eqs. (C10) and (C11) one of the end-points, namely the point of infinity, is harmless, as the integrands in both expressions are vanishing at this point; by choosing  $\mu = (\mu_N + \mu_{N+1})/2$  and  $\Delta = (\mu_{N+1} - \mu_N)/2$ , owing to the assumption of continuous-differentiability with respect to  $k$  of  $\Sigma(k; \mu_N)$  and  $\Sigma(k; \mu_{N+1})$  in a neighbourhood of  $k = k_F$ , a possible divergence of  $S_m^{(0+)}(k)$  or  $\partial S_m^{(0+)}(k)/\partial k$ ,  $m = 0, 1$ , at  $k = k_F$  *cannot* be ascribed to an end-point singularity.

$k = k_F$  must therefore be compensated by a counter contribution arising from  $S_0^{(0+)}(k)$ . With reference to our above discussions, in such an event  $\partial\{\tilde{\Sigma}(k; z) + \tilde{\Sigma}(k; 2\mu - z) - 2\Sigma^{HF}(k)\}/\partial k$  must be singular at  $k = k_F$  for *all*  $z$ ,  $\text{Im}(z) \neq 0$  (see specifically footnote 62). For the special choice of  $z = \mu + \Delta$ , with  $\mu = (\mu_N + \mu_{N+1})/2$  and  $\Delta = (\mu_{N+1} - \mu_N)/2$ ,  $\tilde{\Sigma}(k; z)$  and  $\tilde{\Sigma}(k; 2\mu - z)$  are continuously differentiable functions of  $k$  in a neighbourhood of  $k = k_F$  (by the Fermi-liquid assumption); nonetheless, in accordance with our expectation  $\partial\{\tilde{\Sigma}(k; z) + \tilde{\Sigma}(k; 2\mu - z) - 2\Sigma^{HF}(k)\}/\partial k$  remains divergent at  $k = k_F$  owing to the assumed divergence of  $\partial\Sigma^{HF}(k)/\partial k$  at  $k = k_F$ . On the other hand, for an arbitrary  $z$  the continuous differentiability with respect to  $k$  of  $\tilde{\Sigma}(k; z)$  and  $\tilde{\Sigma}(k; 2\mu - z)$  is not guaranteed, so that divergence of  $\partial\{\tilde{\Sigma}(k; z) + \tilde{\Sigma}(k; 2\mu - z) - 2\Sigma^{HF}(k)\}/\partial k$  at  $k = k_F$  can be in part due to that of  $\partial\{\tilde{\Sigma}(k; z) + \tilde{\Sigma}(k; 2\mu - z)\}/\partial k$  at  $k = k_F$ . In view of the asymptotic relation in Eq. (30) (see footnote 44), it follows that  $\beta_k$  and  $\alpha_k$  may *not* be necessarily continuously differentiable at  $k = k_F$ . The divergence of  $\partial\beta_k/\partial k$  as  $k \rightarrow k_F^\mp$  gives rise to a divergent  $\partial n_s(k)/\partial k$ , as well as a divergent  $\partial n(k)/\partial k$ , at  $k = k_F^\mp$  (for a discussion of this case see § VI). Since for Fermi liquids (here as characterised by  $Z_{k_F} \neq 0$ ), a divergent  $\partial n(k)/\partial k$ , at  $k = k_F^\mp$ , has proved possible (Belyakov 1961, Sartor and Mahaux 1980), we observe that in general even for Fermi liquids  $\Sigma(k; \varepsilon)$  is *not* a continuously differentiable function of  $k$  (in a neighbourhood of  $k = k_F$ ) for  $\varepsilon \neq \varepsilon_F^\mp$ ; at  $\varepsilon = \varepsilon_F^\mp$ ,  $\beta_k$  and  $\alpha_k$  are excluded from contributing to  $\Sigma(k; \varepsilon)$  (see Eq. (30)).

## 2. Marginal Fermi liquids

Having considered the case of isotropic Fermi liquids in considerable detail, below we briefly deal with the case of the isotropic marginal Fermi liquids.

For marginal Fermi liquids we have  $\Sigma''(k; \varepsilon) \equiv \text{Im}\Sigma(k; \varepsilon) \sim -\alpha_k(\varepsilon - \varepsilon_F)$ , as  $\varepsilon \rightarrow \varepsilon_F$ , with  $\alpha_k \geq 0$  (see § V). For the  $\Sigma'_*(k; \varepsilon)$  corresponding to this  $\Sigma''(k; \varepsilon)$  one obtains an expression which except for the first term is identical with that presented in Eq. (C6). For this first term, which we denote by  $\Sigma'_{*,1}(k; \varepsilon)$ , we have (as in the Fermi-liquid case, below  $\Delta > |\delta\varepsilon|$ )

$$\Sigma'_{*,1}(k; \varepsilon) := \frac{-2\alpha_k\delta\varepsilon}{\pi} \mathcal{P} \int_0^\Delta d\varepsilon' \frac{\varepsilon'}{\varepsilon'^2 - \delta\varepsilon^2} \equiv \frac{-2\alpha_k\delta\varepsilon}{\pi} \left\{ \ln(\Delta) - \ln|\delta\varepsilon| - \frac{1}{2\Delta^2} \delta\varepsilon^2 \right\}; \quad (\text{C12})$$

(see Gradshteyn and Ryzhik 1965, p. 59). One observes that here, contrary to the Fermi-liquid case, the limit  $\Delta \downarrow 0$  cannot be taken. This is related to the fact that here  $\Sigma''(k; \pm\varepsilon' + \varepsilon_F) \sim \varepsilon'$  for  $\varepsilon' \rightarrow 0$ , so that *only*  $S_0^{(\Delta)}(k)$  has a limit for  $\Delta \downarrow 0$ . Therefore, in the present case, only the first expression in Eq. (C9) is meaningful. Further, from Eq. (C12) we observe that the *leading* asymptotic contribution to  $\Sigma'_*(k; \varepsilon)$ , as  $\varepsilon \rightarrow \varepsilon_F$ , involves  $\ln|\delta\varepsilon|$  which is singular for  $\delta\varepsilon = 0$ . Combining the above results, we have  $\Sigma(k; \varepsilon) \sim \Sigma(k; \varepsilon_F) + \beta_k(\varepsilon - \varepsilon_F) \ln|\varepsilon - \varepsilon_F| + [\gamma_k + i\alpha_k](\varepsilon - \varepsilon_F)$ , as  $\varepsilon \rightarrow \varepsilon_F$ , where

$$\beta_k \equiv \frac{2\alpha_k}{\pi}, \quad (\text{C13})$$

$$\gamma_k \cong -\frac{2\alpha_k}{\pi} \ln(\Delta) + S_1^{(\Delta)}(k). \quad (\text{C14})$$

The first term on the right-hand side of Eq. (C14) cancels the logarithmically divergent contribution arising from the second term as  $\Delta \downarrow 0$ . It can be readily verified that to the order in which  $-\alpha_k(\varepsilon - \varepsilon_F)$  is an exact representation of  $\Sigma''(k; \varepsilon)$  for  $\varepsilon \rightarrow \varepsilon_F$ ,  $\partial\gamma_k/\partial\Delta = 0$ . This invariance property correctly reflects the fact that  $\gamma_k$  must not depend on the cut-off energy  $\Delta$ .

## APPENDIX D: SINGLE-PARTICLE GREEN FUNCTION AND SELF-ENERGY OF THE LUTTINGER MODEL; SOME ASYMPTOTIC EXPRESSIONS

In this Appendix we consider the single-particle Green function and the self-energy pertaining to the Luttinger model (Luttinger 1963, Mattis and Lieb 1965) for spin-less fermions in some detail.<sup>63</sup> From the expression for the

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<sup>63</sup> We note that an explicit expression for the single-particle Green function for the Luttinger model with  $\delta$ -function interaction in real space (this model is equivalent with the 1 + 1 Thirring (1958) model, dealt with by Johnson (1961)) has been evaluated

former function we can explicitly test the validity of one of our findings in Appendix C, namely that when  $\partial\tilde{\Sigma}(k; z)/\partial k$  is divergent at  $k = k_0$  and  $z = z_0$ , it is divergent at  $k = k_0$  for *all*  $z$ . Further, we consider the asymptotic behaviour of both  $G(k; \varepsilon)$  and  $\Sigma(k; \varepsilon)$  for  $\varepsilon$  in the close vicinity of  $\varepsilon_F$ . In doing so we separately deal with the case where  $k = k_F$  and where  $k \neq k_F$ . In order to remain close to the available literature concerning the Luttinger model, in this Appendix we adopt the commonly-used notations and units and therefore in these deviate from the other parts of the present work. We specify the new notations as we proceed through this Appendix. In this Appendix  $\hbar = 1$ .

### 1. The single-particle Green function and its derivative with respect to momentum

We first construct the single-particle Green function  $G$  from the spectral function  $\rho$  corresponding to the retarded part  $G^R$  of the Green function. For this spectral function we have (Voit 1993b, 1993a)

$$\begin{aligned} \rho_r(q, \omega) &= \frac{\Lambda}{2v_0\Gamma^2(\gamma_0)} \Theta(\omega + rv_0q) \Theta(\omega - rv_0q) \gamma\left(\gamma_0, \frac{\Lambda}{2v_0}(\omega + v_0rq)\right) \\ &\quad \times \left(\frac{\Lambda}{2v_0}(\omega - v_0rq)\right)^{\gamma_0-1} \exp\left(-\frac{\Lambda}{2v_0}(\omega - v_0rq)\right) \\ &\quad + (\omega \rightarrow -\omega, q \rightarrow -q), \quad \text{for } \gamma_0 \neq 0, \\ \rho_r(q, \omega) &\equiv \rho_{0;r}(q, \omega) := \delta(\omega - v_0rq), \quad \text{for } \gamma_0 = 0. \end{aligned} \quad (\text{D1})$$

Here  $r = \mp$  specifies the left and right branches respectively of the single-particle spectrum in the Luttinger model,  $\omega \equiv \varepsilon - \mu$  and  $k \equiv q + rk_F$ . It is important to note that contrary to the 2D and 3D cases, *here  $k$  and  $q$  take on both positive and negative values. Moreover  $q$  is measured with respect to  $rk_F$ ;  $k$ , on the other hand, is measured with respect to the origin.* Further,  $\Lambda$  stands for the (finite) cut-off on the range of the interaction in the momentum space — the precise value of  $\Lambda$  is not of relevance here (Sólyom 1979; see §§ 9 and 2 herein);  $\Gamma(z)$  stands for the Gamma function (Abramowitz and Stegun 1972, p. 255),  $\gamma_0 \equiv \alpha/2$ , with  $\alpha$  the ‘anomalous dimension’ (see § IV.C) and  $\gamma(a, z)$  denotes the incomplete Gamma function (Abramowitz and Stegun 1972, p. 260). Unless we explicitly indicate otherwise, below  $0 < \alpha < 1$  and thus  $0 < \gamma_0 < 1/2$ . With

$$\rho_r(q, \omega) := -\frac{1}{\pi} \text{Im} G_r^R(q, \omega), \quad (\text{D3})$$

where the  $G_r^R$  denotes the retarded Green function, making use of the fact that  $\text{Im} G_r(q, \omega) = \text{sgn}(\omega) \text{Im} G_r^R(q, \omega)$ , we have

$$G_r(q, \omega) = \int_{-\infty}^{\infty} d\omega' \frac{\rho_r(q, \omega')}{\omega - \omega' + i\eta \text{sgn}(\omega')}, \quad \eta \downarrow 0. \quad (\text{D4})$$

From Eq. (26), introducing  $\omega_{r;q}^0 := rv_0q$ , we obtain (compare with Eq. (33))<sup>65</sup>

$$\frac{\partial \tilde{G}_r(q, z)}{\partial q} = \tilde{G}_r^2(q, z) \left\{ \frac{\partial \omega_{r;q}^0}{\partial q} + \frac{\partial \tilde{\Sigma}_r(q, z)}{\partial q} \right\}. \quad (\text{D5})$$

Since  $\tilde{G}_r(q, z)$  is an entire function of  $z$  for all  $z$ ,  $\text{Im}(z) \neq 0$  (see § II) and, since  $\partial \omega_{r;q}^0/\partial q \equiv rv_0$  is finite, any possible divergence of  $\partial \tilde{G}_r(q, z)/\partial q$ , as function of  $q$  and  $z$  ( $\text{Im}(z) \neq 0$ ), must be due to a divergence of  $\partial \tilde{\Sigma}_r(q, z)/\partial q$  on the

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by Theumann (1967). The corresponding expression in Eq. (2.19)<sub>T</sub> (here subscript  $T$  denotes an equation from Theumann’s work) turns out to be identically vanishing for the case of non-interacting Luttinger model. To verify this, note that the coupling constant  $\lambda$  of interaction as presented in Eq. (1.4)<sub>T</sub>, determines  $\Omega$  in Eq. (2.3)<sub>T</sub> which in turn fixes  $\beta$  in Eq. (2.14)<sub>T</sub>. It is seen that for non-interacting particles, corresponding to  $\lambda = 0$  and thus  $\beta = 1$ ,  $G_1(k, \omega)$  is identically vanishing, owing to  $\sin \pi\beta$  in Eq. (2.19)<sub>T</sub>. We have not attempted to identify the origin of this shortcoming of  $G_1(k, \omega)$  in Eq. (2.19)<sub>T</sub>.

<sup>64</sup> Thus a better notation for  $q$  would be  $q_r$  in order to emphasise this  $r$ -dependence of the origin.

<sup>65</sup> It is readily verified that  $\tilde{G}_r(q, z) \equiv \int_{-\infty}^{\infty} d\omega' \rho_r(q, \omega')/(z - \omega')$ , for  $\text{Im}(z) \neq 0$ ; in view of Eq. (5),  $G_r(q, \omega) = \lim_{\eta \downarrow 0} \tilde{G}_r(q, \omega \pm i\eta)$ , for  $\omega \gtrless 0$ , which can be easily shown to coincide with that presented in Eq. (D4). We point out that, with  $\rho_{0;r}(q, \omega) = \delta(\omega - \omega_{r;q}^0)$  (see Eq. (D2)), one obtains  $G_{0;r}(q, \omega) = 1/[\omega - \omega_{r;q}^0 + i\eta \text{sgn}(\omega_{r;q}^0)]$  and consequently  $\tilde{G}_{0;r}(q, z) = 1/[z - \omega_{r;q}^0]$ , for  $\text{Im}(z) \neq 0$ .

right-hand side of Eq. (D5). In order to be able to evaluate  $\partial\tilde{G}_r(q, z)/\partial q$ , which according to Eq. (D4) involves  $\partial\rho_r(q, \omega)/\partial q$ , some changes in the expression on the right-hand side of Eq. (D1) are necessary. To appreciate this, note that  $\partial\Theta(\omega - rv_0q)/\partial q = -rv_0\delta(\omega - rv_0q)$ , from which, on account of the term  $(\omega - rv_0q)^{\gamma_0-1}$ , one immediately observes that independent of the value of  $q$ ,  $\partial\rho_r(q, \omega)/\partial q = \infty$ . This would imply a peculiar situation where  $\rho_r(q, \omega)$ , which is continuous almost everywhere, would be *nowhere* on the  $q$ -axis differentiable. As we shall see, this is wholly attributable to the fact that the expressions on the right-hand side of Eq. (D1) which are multiplied by *distributions*  $\Theta(\omega \mp rv_0q)$  and  $\Theta(-\omega \pm rv_0q)$  do *not* qualify as *test functions* (Gelfand and Shilov 1964). For simplicity, in the following we shall explicitly deal with the case corresponding to  $r = +$ . We write

$$\rho_+(q, \omega) \equiv \Theta(\omega + v_0q)\Theta(\omega - v_0q)\phi(q, \omega) + (\omega \rightarrow -\omega, q \rightarrow -q), \quad (D6)$$

$$\phi(q, \omega) := \mathcal{A}_1 \phi_1(\omega + v_0q) \phi_2(\omega - v_0q), \quad (D7)$$

$$\mathcal{A}_\sigma := \frac{1}{\Gamma^2(\gamma_0)} \left( \frac{\Lambda}{2v_0} \right)^\sigma, \quad (D8)$$

$$\phi_1(x) := \gamma \left( \gamma_0, \frac{\Lambda}{2v_0} x \right), \quad (D9)$$

$$\phi_2(x) := \left( \frac{\Lambda}{2v_0} x \right)^{\gamma_0-1} \exp \left( - \frac{\Lambda}{2v_0} x \right). \quad (D10)$$

Since  $0 < \gamma_0 < 1/2$ , it is evident that the problem at hand (i.e.  $\partial\rho_+(q, \omega)/\partial q = \infty$ ) has its origin in the singularity of  $\phi_2(x)$  at  $x = 0$ . Through application of the prescription as specified in Fig. 3 (see caption to this Figure), we define  $\tilde{G}_{+, \nu}(q, z)$  and consider  $\tilde{G}_+(q, z)$  as being obtained through  $\lim_{\nu \downarrow 0} \tilde{G}_{+, \nu}(q, z)$ . We have

$$\begin{aligned} \tilde{G}_{+, \nu}(q, z) \equiv & \int_{-\infty}^{-v_0q} d\omega' \frac{\phi(-q, -\omega')}{z - \omega'} + \int_{v_0q+\nu}^{\infty} d\omega' \frac{\phi(q, \omega')}{z - \omega'} \\ & + \int_{v_0q-\nu}^{v_0q} d\omega' \frac{\phi(q, v_0q + \nu) [1 + (\omega' - v_0q)/\nu]}{z - \omega'} + \int_{v_0q}^{v_0q+\nu} d\omega' \frac{\phi(q, v_0q + \nu)}{z - \omega'}. \end{aligned} \quad (D11)$$

It is from this expression that the derivative with respect to  $q$  has to be taken; we then have  $\partial\tilde{G}_+(q, \omega)/\partial q := \lim_{\nu \downarrow 0} \partial\tilde{G}_{+, \nu}(q, \omega)/\partial q$ . After some algebra, one obtains the following final result

$$\begin{aligned} \frac{\partial\tilde{G}_+(q, z)}{\partial q} = & -v_0 \frac{\phi(-q, v_0q)}{z + v_0q} + \int_{-\infty}^{-v_0q} d\omega' \frac{\partial\phi(-q, -\omega')/\partial q}{z - \omega'} \\ & + \mathcal{A}_1 v_0 \int_{v_0q}^{\infty} d\omega' \frac{\{\partial\phi_1(\omega' + v_0q)/\partial\omega'\} \phi_2(\omega' - v_0q)}{z - \omega'} \\ & + \mathcal{A}_1 v_0 \int_{v_0q}^{\infty} d\omega' \phi_2(\omega' - v_0q) \frac{\partial}{\partial\omega'} \left\{ \frac{\phi_1(\omega' + v_0q)}{z - \omega'} \right\}. \end{aligned} \quad (D12)$$

In arriving at this expression, we have made use of the properties  $\partial\phi_{1,2}(\omega' \pm v_0q)/\partial q = \pm v_0 \partial\phi_{1,2}(\omega' \pm v_0q)/\partial\omega'$ . It is evident that the integrands of the last two integrals on the right-hand side of Eq. (D12) are singular (owing to  $\phi_2(\omega' - v_0q)$ ) at the lower limits of the integration boundaries; however since  $0 < \gamma_0 < 1/2$ , these singularities are integrable and therefore  $\partial\tilde{G}_+(q, z)/\partial q$  as presented in Eq. (D12) is a well-defined function of  $q$  and  $z$ .

We are now in a position to draw the important conclusion that since  $\phi(-q, v_0q) = \mathcal{A}_1 \phi_1(0) \phi_2(2v_0q) \propto q^{\gamma_0-1}$ ,  $\partial\tilde{G}_+(q, z)/\partial q$  diverges as  $q \downarrow 0$ , for *all*  $z$ . This is in full conformity with our statement in Appendix C that, when  $\partial\tilde{G}_+(q, z)/\partial q$  is unbounded at a particular value of  $q$ , it is unbounded at that  $q$  for *all*  $z$ .

## 2. Self-energy and its asymptotic forms close to the Fermi points and the Fermi energy

Now we proceed with evaluating the asymptotic behaviour of  $\Sigma_r(q, \omega)$  for  $\omega \rightarrow 0$ . We consider two cases, corresponding to  $q = 0$  and  $q \neq 0$ . We demonstrate that, as  $\gamma_0$  approaches zero, the self-energy of the Luttinger (1963) model becomes vanishingly small in a manner that is specific to non-interacting Fermi systems.

Since we are interested in cases where  $0 < \gamma_0 < 1/2$ , for the determination of the *leading* asymptotic contribution to  $G_r(q, \omega)$ , and thus to  $\Sigma_r(q, \omega)$ , for  $q \rightarrow 0$  (i.e.  $k \rightarrow k_F$ ) and  $\omega \rightarrow 0$  (i.e.  $\varepsilon \rightarrow \varepsilon_F$ ) we need to merely take account of the  $\omega'$ -integration in Eq. (D4) restricted to the interval  $[-\Delta, \Delta]$ , where  $\Delta$  is some finite constant satisfying  $|\omega| < \Delta$  (see Appendix C). Thus we define (*cf.* Eq. (D4))

$$\tilde{G}_r^{(\Delta)}(q, z) := \int_{-\Delta}^{\Delta} d\omega' \frac{\rho_r(q, \omega')}{z - \omega'}. \quad (\text{D13})$$

Below we shall deal with  $G_r^{(\Delta)}(q, \omega)$ .

*a. The  $q = 0$  case*

From the expression in Eq. (D1) it can readily be deduced that (see Eqs. (D6)-(D10))

$$\rho_r(q = 0, \omega) \sim \mathcal{A}_{\gamma_0+1} |\omega|^{2\gamma_0-1}, \quad \text{for } \omega \rightarrow 0. \quad (\text{D14})$$

Making use of the general result ( $|\omega| < \Delta$ )

$$\begin{aligned} \mathcal{P} \int_0^{\Delta} d\omega' \frac{\omega'^{\sigma}}{\omega'^2 - \omega^2} &= \frac{\pi}{2} \tan\left(\frac{\pi\sigma}{2}\right) |\omega|^{\sigma-1} + \frac{\Delta^{\sigma-1}}{\sigma-1} {}_2F_1\left(1, \frac{1-\sigma}{2}; \frac{3-\sigma}{2}; \frac{\omega^2}{\Delta^2}\right) \\ &= \frac{\pi}{2} \tan\left(\frac{\pi\sigma}{2}\right) \omega^{\sigma-1} + \frac{\Delta^{\sigma-1}}{\sigma-1} + \frac{\Delta^{\sigma-3}}{\sigma-3} \omega^2 + \mathcal{O}(\omega^4), \\ &\quad \sigma > -1, \sigma \neq 1, 3, \dots, \end{aligned} \quad (\text{D15})$$

where  ${}_2F_1(a, b; c; z)$  stands for the Gauss Hypergeometric function (Abramowitz and Stegun 1972, p. 556), for  $0 < \gamma_0 < 1$  and  $|\omega| < \Delta$  we obtain

$$G_r^{(\Delta)}(q = 0, \omega) \sim -\pi \mathcal{A}_{\gamma_0+1} \left\{ \tan\left(\frac{\pi(2\gamma_0-1)}{2}\right) + i \right\} \text{sgn}(\omega) |\omega|^{2\gamma_0-1}. \quad (\text{D16})$$

Evidently, for  $0 < \gamma_0 < 1/2$ , the right-hand side of this expression diverges as  $\omega \rightarrow 0$ . As a consequence, for  $0 < \gamma_0 < 1/2$ , the possibly non-vanishing constant  $\{G_r(q = 0, \omega) - G_r^{(\Delta)}(q = 0, \omega)\}$  is asymptotically irrelevant for  $\omega \rightarrow 0$  and we can write

$$G_r(q = 0, \omega) \sim \pi \mathcal{A}_{\gamma_0+1} \left\{ \cot(\pi\gamma_0) - i \right\} \text{sgn}(\omega) |\omega|^{2\gamma_0-1}, \quad \omega \rightarrow 0 \quad (0 < \gamma_0 < 1/2). \quad (\text{D17})$$

From the Dyson equation we have

$$\Sigma_r(q, \omega) = G_{0;r}^{-1}(q, \omega) - G_r^{-1}(q, \omega). \quad (\text{D18})$$

Since  $G_{0;r}(q, \omega) = 1/[\omega - v_0 r q + i \text{sgn}(v_0 r q)]$  (see footnote 65), it follows that

$$G_{0;r}^{-1}(q, \omega) = \omega - v_0 r q, \quad (\text{D19})$$

and thus  $G_{0;r}^{-1}(q = 0, \omega) = \omega$ . For  $0 < \gamma_0 < 1$ , the leading-order contribution to  $\Sigma_r(q = 0, \omega)$  is therefore entirely due to  $G_r^{-1}(q = 0, \omega)$  on the right-hand side of Eq. (D18). From Eq. (D16) we thus obtain ( $0 < \gamma_0 < 1/2$ )

$$G_r^{-1}(q = 0, \omega) \sim \frac{1}{\pi \mathcal{A}_{\gamma_0+1} (\cot^2(\pi\gamma_0) + 1)} \left\{ \cot(\pi\gamma_0) + i \right\} \text{sgn}(\omega) |\omega|^{1-2\gamma_0}. \quad (\text{D20})$$

As a consequence of our above arguments, for  $0 < \gamma_0 < 1/2$  we have (for  $\gamma_0$  in this range,  $\cot(\pi\gamma_0) > 0$ )

$$\Sigma'_r(q = 0, \omega) \sim \frac{-\cot(\pi\gamma_0) \text{sgn}(\omega)}{\pi \mathcal{A}_{\gamma_0+1} (\cot^2(\pi\gamma_0) + 1)} |\omega|^{1-2\gamma_0}, \quad (\text{D21})$$

$$\Sigma''_r(q = 0, \omega) \sim \frac{-\text{sgn}(\omega)}{\pi \mathcal{A}_{\gamma_0+1} (\cot^2(\pi\gamma_0) + 1)} |\omega|^{1-2\gamma_0}. \quad (\text{D22})$$

We note that  $\Sigma_r(q = 0, \omega = 0) = 0$  (see Eq. (D17) and text preceding it), which in view of our considerations in Appendix B amounts to the fact that in the Luttinger model  $\Sigma_r^{HF}(q = 0)$  is entirely cancelled by correlation effects. It is further interesting to note that

$$\frac{\Sigma''_r(q = 0, \omega)}{\Sigma'_r(q = 0, \omega) - \Sigma_r(q = 0, \omega = 0)} \sim \tan(\pi\gamma_0), \quad \omega \rightarrow 0, \quad (\text{D23})$$

which is seen to approach zero (like  $\sim \pi\gamma_0$ ) for  $\gamma_0 \downarrow 0$ . As we have mentioned in § IV.C,  $\gamma_0 = 0$  corresponds to the non-interacting system, so that the expressions presented in Eqs. (D21) and (D22) reproduce the non-interacting limit in a continuous way.

b. The  $q \neq 0$  case

Here we confine our considerations to the case corresponding to  $r = +$ . From Eq. (D11) (taking the limit  $\nu \downarrow 0$ ) it readily follows that for  $0 < \gamma_0 < 1$  the leading asymptotic contribution to  $G_+(q, \omega)$ , as  $\omega \rightarrow 0$ , is due to

$$\tilde{\mathcal{G}}_+^{(\Delta)}(q, z) := \int_{v_0 q}^{v_0 q + \Delta} d\omega' \frac{\phi(q, \omega')}{z - \omega'}. \quad (\text{D24})$$

For  $z = v_0 q + \omega + i\eta$ ,  $\eta \downarrow 0$ , with  $0 < \omega < \Delta$ , making use of  $\gamma(\gamma_0, \Lambda(\omega' + 2v_0 q)/[2v_0]) = \gamma(\gamma_0, \Lambda q) + \mathcal{O}(\omega')$ ,  $\exp(-\Lambda\omega'/[2v_0]) = 1 + \mathcal{O}(\omega')$ , while employing the standard result ( $0 < \omega < \Delta$ )

$$\begin{aligned} \mathcal{P} \int_0^\Delta d\omega' \frac{\omega'^{\sigma-1}}{\omega' - \omega} &= -\pi \cot(\pi\sigma) \omega^{\sigma-1} + \frac{\Delta^{\sigma-1}}{\sigma-1} {}_2F_1\left(1, 1-\sigma; 2-\sigma; \frac{\omega}{\Delta}\right) \\ &= -\pi \cot(\pi\sigma) \omega^{\sigma-1} - \frac{\Delta^{\sigma-1}}{1-\sigma} - \frac{\Delta^{\sigma-2}}{2-\sigma} \omega + \mathcal{O}(\omega^2), \\ &\quad \sigma > 0, \sigma \neq 1, 2, \dots, \end{aligned} \quad (\text{D25})$$

we obtain ( $0 < \omega < \Delta$ ,  $0 < \gamma_0 < 1$  and  $q \neq 0$ )

$$\mathcal{G}_+^{(\Delta)}(q, \omega) \sim G_+(q, \omega) \sim \pi \gamma(\gamma_0, \Lambda q) \mathcal{A}_{\gamma_0} \left\{ \cot(\pi\gamma_0) - i \right\} \omega^{\gamma_0-1}, \quad \omega \downarrow 0. \quad (\text{D26})$$

Note that the exponent  $\gamma_0 - 1$ , as compared with  $2\gamma_0 - 1$  for the case corresponding  $q = 0$ , implies that even for  $\gamma_0$  in the open interval  $(0, 1)$  (as opposed to  $(0, 1/2)$ ),  $G_+(q \neq 0, \omega)$  is to leading order asymptotically identical with the corresponding auxiliary function  $\mathcal{G}_+^{(\Delta)}(q \neq 0, \omega)$  as  $\omega \downarrow 0$ ; recall that the leading asymptotic terms of  $G_r^{(\Delta)}(q = 0, \omega)$  and  $G_r(q = 0, \omega)$  for  $\omega \rightarrow 0$  (see Eqs. (D16) and (D17)) are identical only when  $0 < \gamma_0 < 1/2$ . Further, similar to the case corresponding to  $q = 0$ , the divergence of  $G_+(q, \omega)$  for  $0 < \gamma_0 < 1$ , as  $\omega \downarrow 0$ , implies that the possibly non-vanishing constant  $\{G_+(q \neq 0, \omega) - \mathcal{G}_+^{(\Delta)}(q \neq 0, \omega)\}$  is asymptotically irrelevant.

Proceeding as in the case of  $q = 0$ , for  $\omega \downarrow 0$  we obtain ( $0 < \gamma_0 < 1$ )

$$G_+^{-1}(q, \omega) \sim \frac{1}{\pi \gamma(\gamma_0, \Lambda q) \mathcal{A}_{\gamma_0} (\cot^2(\pi\gamma_0) + 1)} \left\{ \cot(\pi\gamma_0) + i \right\} \omega^{1-\gamma_0}. \quad (\text{D27})$$

As a consequence of our above arguments, for  $0 < \gamma_0 < 1$  we have (for  $\gamma_0$  in this range,  $\cot(\pi\gamma_0)$  takes on both positive as well as negative values)

$$\Sigma'_+(q, \omega) \sim \frac{-\cot(\pi\gamma_0)}{\pi \gamma(\gamma_0, \Lambda q) \mathcal{A}_{\gamma_0} (\cot^2(\pi\gamma_0) + 1)} \omega^{1-\gamma_0}, \quad (\text{D28})$$

$$\Sigma''_+(q, \omega) \sim \frac{-1}{\pi \gamma(\gamma_0, \Lambda q) \mathcal{A}_{\gamma_0} (\cot^2(\pi\gamma_0) + 1)} \omega^{1-\gamma_0}. \quad (\text{D29})$$

We point out that  $\Sigma_+(q, \omega = 0) = 0$  (see Eq. (D17) and text preceding it), which in view of our considerations in Appendix B leads to the conclusion that, in the Luttinger model  $\Sigma_+^{HF}(q)$  is entirely cancelled by correlation effects. As in the case corresponding to  $q = 0$ , here we have (compare with Eq. (D23))

$$\frac{\Sigma''_+(q, \omega)}{\Sigma'_+(q, \omega) - \Sigma_+(q, \omega = 0)} \sim \tan(\pi\gamma_0), \quad \omega \downarrow 0, \quad (\text{D30})$$

which approaches zero (like  $\sim \pi\gamma_0$ ) for  $\gamma_0 \downarrow 0$ ; this result is consistent with the fact that  $\gamma_0 = 0$  corresponds to the non-interacting system.

We further point out that since  $\partial\gamma(a, z)/\partial z = \exp(-z)z^{a-1}$ , from Eqs. (D28) and (D29) we obtain that  $\partial\Sigma_+(q, \omega)/\partial q \propto |q|^{\gamma_0-1}$  which diverges as  $q \rightarrow 0$  for  $\gamma_0 < 1$ . This result is in full conformity with our finding above (see the last paragraph in § D.1, following Eq. (D12)).

In closing this Appendix, we point out that the apparent difference in the asymptotic behaviours of  $\Sigma_r(q, \omega)$  (specifically concerning  $r = +$  explicitly considered here) for  $q \rightarrow 0$  and  $q = 0$ , is *not* specific to the Luttinger model. It is well known (Hodges, Smith and Wilkins 1971, Bloom 1975, Fujimoto 1990, Fukuyama, Narikiyo and Hasegawa 1991, Fukuyama, Hasegawa and Narikiyo 1991) that for isotropic systems of interacting electrons in two



spatial dimensions, under the assumption that they are Fermi liquids, while  $\Sigma''(k; \varepsilon) \sim'(\varepsilon - \varepsilon_F)^2$  for  $k \neq k_F$ , we have  $\Sigma''(k_F; \varepsilon) \sim'(\varepsilon - \varepsilon_F)^2 \ln |\varepsilon - \varepsilon_F|$ .  $\square$

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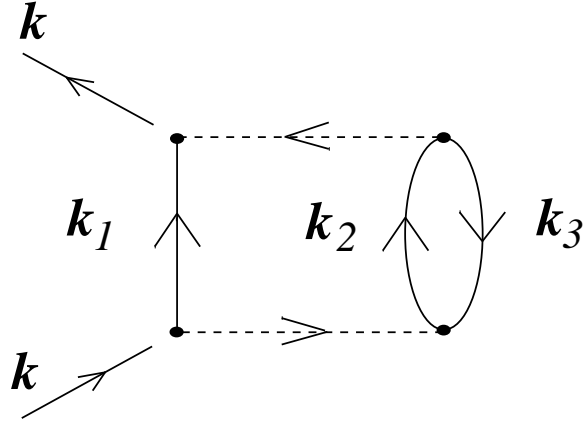


FIG. 1. A second-order skeleton self-energy diagram in terms of the bare particle-particle interaction  $v$  (broken lines) employed by Luttinger (1961). The solid lines stand for either  $G_0$  or  $G$ , the single-particle Green function corresponding to the non-interacting and interacting systems, respectively. The external and internal wave-vectors,  $\mathbf{k}$  and  $\mathbf{k}_1$ ,  $\mathbf{k}_2$  and  $\mathbf{k}_3$ , respectively, are shown. Because of conservation of momentum, one of the internal wave-vectors, say  $\mathbf{k}_3$ , can be eliminated;  $\mathbf{k}_3 = \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}$ .

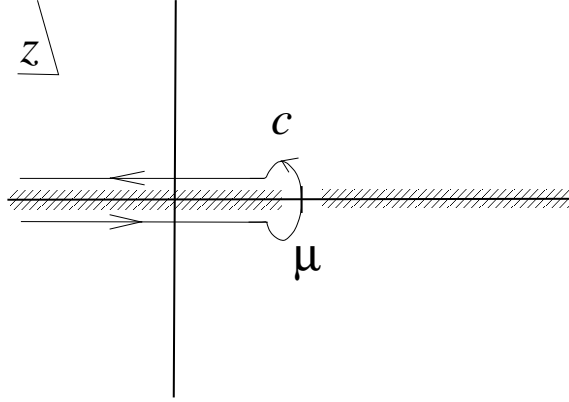


FIG. 2. Contour  $\mathcal{C}$  of integration employed in the proof of the Migdal theorem concerning the discontinuity of the momentum distribution function  $n(k)$  at  $k = k_F$  and its relation to the quasi-particle weight on the Fermi surface,  $Z_{k_F}$ . The shaded parts of the real axis indicate branch cuts of  $\tilde{G}(k; z)$ ; the unshaded part of this axis centred around the ‘chemical potential’  $\mu$  represents  $(\mu_N, \mu_{N+1})$  with  $\mu_N$  and  $\mu_{N+1}$  two branch points of  $\tilde{G}(k; z)$  as well as  $\tilde{\Sigma}(k; z)$ .

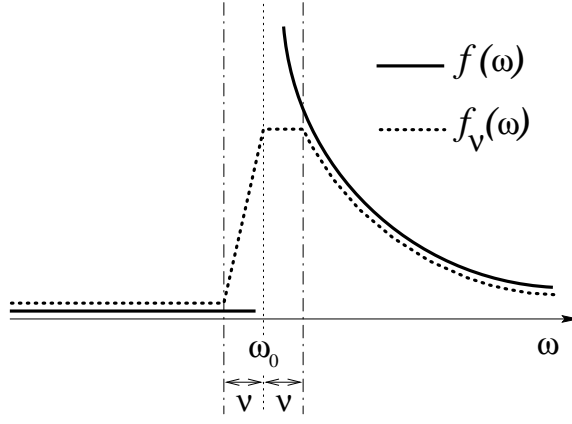


FIG. 3. The *distribution* (as opposed to *function*)  $f(\omega) := \Theta(\omega - \omega_0)g(\omega)$  and a specific function  $f_\nu(\omega)$ , corresponding to a finite positive value of  $\nu$ , in terms of which we choose  $f(\omega)$  to be defined (this choice is not unique);  $f(\omega) := \lim_{\nu \downarrow 0} f_\nu(\omega)$ . We have the following  $f_\nu(\omega) \equiv 0$ , for  $\omega \leq \omega_0 - \nu$ ;  $f_\nu(\omega) \equiv g(\omega_0 + \nu)[1 + (\omega - \omega_0)/\nu]$ , for  $\omega_0 - \nu \leq \omega \leq \omega_0$ ;  $f_\nu(\omega) \equiv g(\omega_0 + \nu)$ , for  $\omega_0 \leq \omega \leq \omega_0 + \nu$  and  $f_\nu(\omega) \equiv g(\omega)$ , for  $\omega \geq \omega_0 + \nu$ . The form of  $f(\omega)$  is reminiscent of  $\rho_+(q, \omega)$  pertaining to the Luttinger model for which  $\omega_0$  coincides with  $v_0 q$  and  $\rho_+(q, \omega) \propto (\omega - v_0 q)^{\gamma_0 - 1}$  for  $\omega \downarrow v_0 q$  (here  $0 < \gamma_0 < 1/2$ ).